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Thesis

Simple Correlation Including the Correlation Coefficient, Correlation from Ranks, and Mean Square Contingency.

by

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OUTLINE

Chapter I A General Discussion of Correlation

A. A General Definition of Correlation

When a series of measures in statistics is studied, it is, in general, desirable to determine three points: (1) the computation of some average, the mean, median, or mode to represent the series; (2).the picturing of the degree of concentration by obtaining a measure of the dispersion; (3) the graphic picture of the distribution by plotting the smoothed frequency curve. When two or more series are to be compared, it is often necessary to find some method of determining the relationship between the series. This relationship is called correlation.

Suppose we consider a hypothetical case in discussing the correlation between marks given to a class of twenty pupils in plane geometry and English.

School Marks Given a Class of Twenty Pupils in Plane Geometry and English

Pupils	Average Marks in Plane Geometry	Average Marks in English	Rank in Achievement in Plane Geometry	Rank in Achievement in English
A	50	60	20	18
B	68	72	17	14
C	92	85	4	8
D	84	91	7	5
E	97	96	2	2
F	72	80	15	10
G	82	75	9	12
H	76	84	13	9
I	62	50	18	20
J	56	55	19	19
K	85	90	6	6
L	98	97	1	1
M	90	79	5	11
N	70	61	16	17
O	83	92	8	4
P	80	74	11	13
Q	96	95	3	3
R	75	71	14	15
S	81	86	10	7
T	78	70	12	16

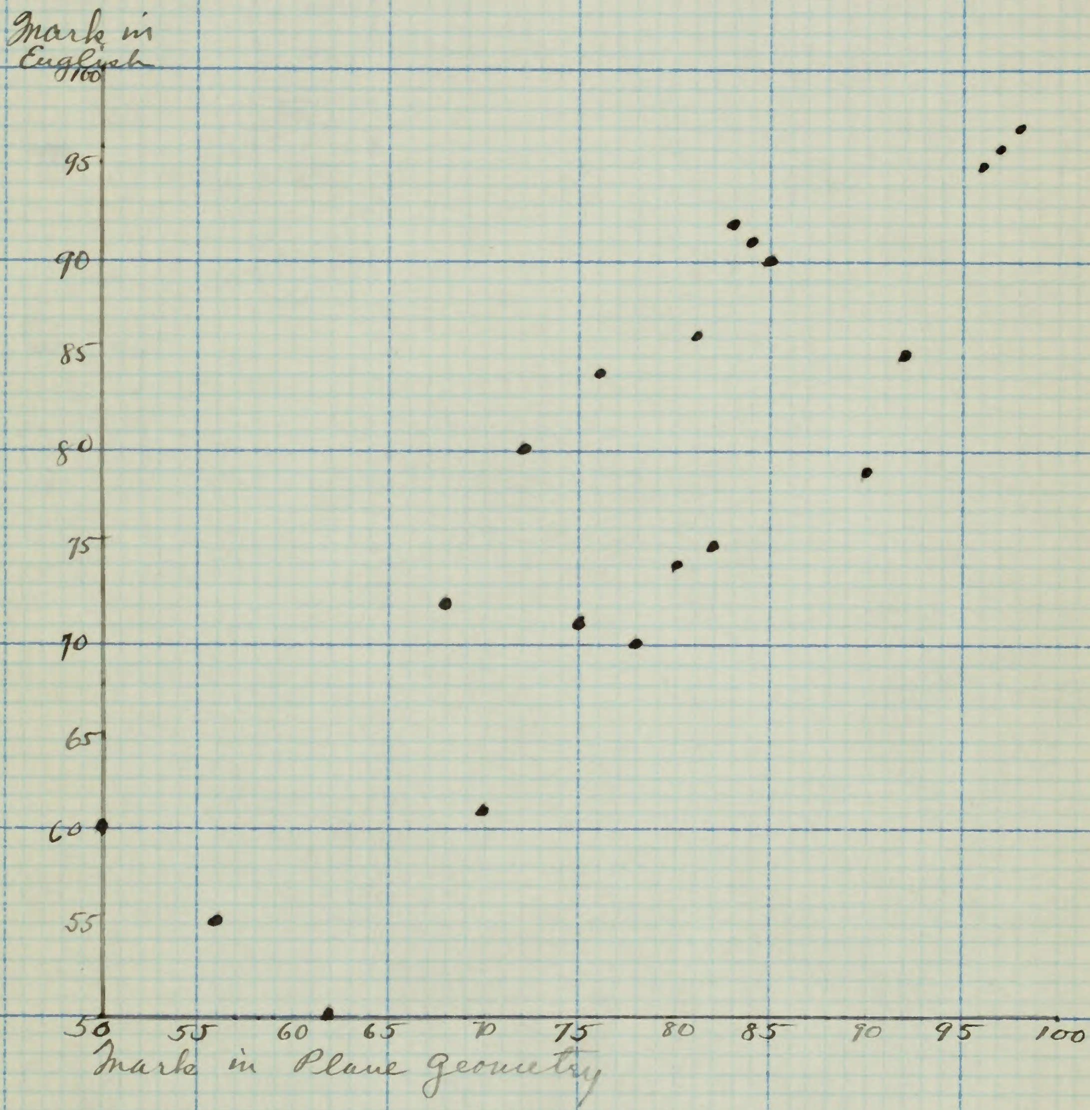


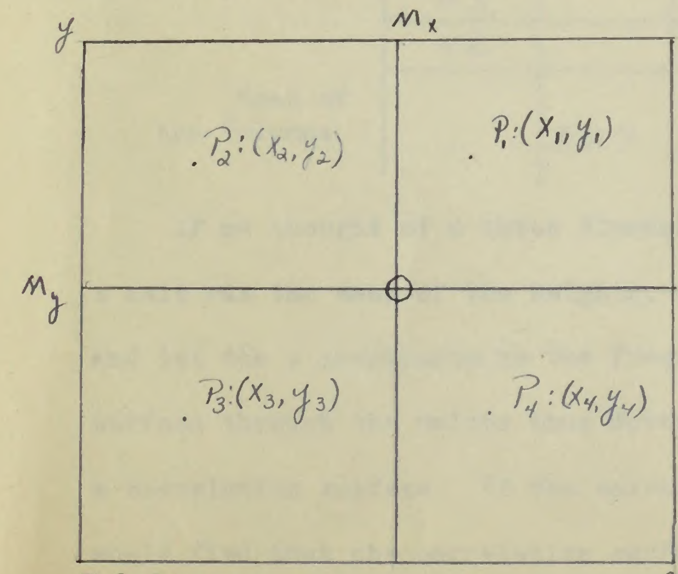
Fig I. Scatter Diagram Showing School Marks Given a Class of Twenty Pupils in Plane Geometry and English

The graph would show readily that a pupil who stood high in plane geometry would also receive high marks in English and if low in geometry was also low in English. This graphic method would give a fair idea of the relationship existing between the two series but would not give an exact numerical expression nor would it give an expression which would summarize the situation.

The problem, then, is to find some device which would yield a numerical expression that would completely describe the relation existing between the two series. This numerical expression is called the "coefficient of correlation." The coefficient of correlation is the term used generally in statistics to refer to the one obtained by the product-moment method and is designated by "r". It is an index of linear correlation which type will be discussed in this paper.

The series in Fig. I form an example of linear correlation because the points tend to form a straight band across the graph. If this were perfect linear correlation, the points would lie on a straight line. Thus we might define correlation as the "tendency for two observed variables to be related in the form of a single-valued mathematical function."

The product-moment formula will be developed later, but a brief explanation of it may aid in stating what is meant by correlation. A measure of relationship between the variables might be obtained by considering products of the deviations from the means expressed in terms of the standard deviations. Let $x = \overline{X} - M_x$ and $y = \overline{Y} - M_y$, $\sigma_x = \frac{1}{N}\Sigma x^2$ and $\sigma_y = \frac{1}{N}\Sigma y^2$, then we could sum the N pairs of x products:



$$\frac{x_1}{\sigma_x}\cdot\frac{y_1}{\sigma_y} + \frac{x_2}{\sigma_x}\cdot\frac{y_2}{\sigma_y} + \cdots + \frac{x_n}{\sigma_x}\cdot\frac{y_n}{\sigma_y}$$

Holzinger
"Statistical Methods in Education"

and divide by N. The result $\frac{\Sigma xy}{N\sigma_x\sigma_y}$ is presented by r and is called the product-moment formula for correlation. It will be shown later that r may vary from -1 (perfect negative correlation) thru 0 (lack of correlation) to +1 (perfect positive correlation).

B. <u>The Correlation Table and Correlation Surface.</u>

If we were to consider the following problem to find the coefficient of correlation between the two series,

(1) Heights in inches of Glasgow school boys, ages 4.5 to 5.5 years,

and (2) Weights in pounds of these same boys,

the work would be arranged in a double entry table called a correlation table. In this table the frequencies are thought of as being concentrated at the midpoint of the class intervals; that is, the weights are divided into class intervals as follows:

24-28, 28-33, 34-38, etc., with 26, 31, 36, etc., the midpoints.

Forsyth "Mathematical Analysis of Statistics" P. 219

Height in Inches	Weight in Pounds 26	31	36	41	46	51
31	2					
34	5	15	5			
37	1	18	72	8		
40		5	87	90	7	1
43			4	35	21	5
46			1		2	
Mean of the Columns	33.7	36.2	38.7	40.6	42.5	43.5

If we thought of a three dimensional coordinate system in which the x axis was the mean of the heights, the y axis the mean of the weights, and let the z coordinates be the frequencies, and if we passed a smooth surface through the points thus determined, we would get what is known as a correlation surface. If the correlation table were symmetrical, we would find that the correlation surface was a normal surface; that is, a

bell-shaped surface with the z axis the centroid vertical.

C. Methods of Approach to the Problem of Correlation.

Rietz "Mathematical Statistics" P. 77

There are two methods of approach to the problem of correlation: one is the "regression" method, the other the "correlation surface" method.

The Regression Method.

If we consider associated values of x and y as plotted in a scatter diagram and separate the dots into classes by selecting class intervals dx and dy, the y's corresponding to any class dx are called an array of y's and similarly the values of x corresponding to any interval dy are called an x-array. The regression curve $y = f(x)$ is defined as the locus of the expected value of y in the array which corresponds to an assigned value of x as dx approches zero; that is, the regression curve of y on x is the locus of the means of the arrays of y's as dx approaches zero. Similarly the regression curve of x on y is the locus of the means of the arrays of x's as dy approaches zero. Having found the regression curves of y on x and x on y, we are now interested in the distribution of the values of y whose average we have predicted. This is accomplished by measuring the dispersion of the values of y which correspond to an assigned value of x. In other words, we wish to know the average standard deviation of a row about the line which represents the locus of the means of the rows and also the average standard deviation of a column about the line which represents the locus of the means of the x-arrays.

To illustrate the regression method we might consider a problem of correlating the marks of a class in geometry and of the same class in English. We would first find a means of predicting the mean mark of a sub-group in the geometry class which had received identical marks in English, then we would find a measure to predict the dispersion of such a subgroup.

The Correlation Surface Method

In this method, we attempt to determine the probability, $\phi(x,y)dx\,dy$, that a pair of associated values (x,y) of x and y will fall into the rectangular area bounded by $(x+dx)$ and $(y+dy)$. If m (x) is such that $m(x)dx$ gives, to within infinitesimals of higher order, the probability that any x lies between x and $(x+dx)$ and $n(x,y)dy$ gives the probability that any y taken from the array which corresponds to the x chosen above will lie between y and $(y+dy)$ then the probability that both will happen is

$$\phi(x,y)\,dx\,dy = m(x)\,n(x,y)\,dx\,dy.$$

We are thus able to set up the equation for the frequency surface $z=\phi(x,y)$ and by a study of this to arrive at the coefficient of correlation between x and y.

Chapter II A Definition of Correlation and an Introduction to the Regression Method.

Having given a general idea of the problem we might now define correlation and then indicate how we would approach a solution by way of the regression method.

Definition: (Prof. Dow in a course in Statistics) "A quantity is said to be correlated with another quantity if to any value of the one quantity there exists a probable value of the other quantity and more exactly we shall call x and y correlated if when any particular values of y are selected, the average value of the corresponding x is thereby determined."

If we consider a problem like that given on page 4 and set up a graph in which we plotted only the mean value of the heights corresponding to any given weight, we would have a graph like the one pictured on page 9. The dots represent the mean values of the heights corresponding to the actual values of the weights. The line $O_1 M_y$ cuts the Y axis at the mean of the heights and $O_1 M_x$ cuts the X axis at the mean of the weights. The line CC' is fitted by inspection as being the line of the best fit which corresponds to the actual line of the means. This line serves the purpose of a generalized trend of the points. Since CC' is the line of best fit of the means of the columns, it must pass through O_1 , the mean point of the entire distribution and if B ; (x,y), a point on the line, is taken so that x and y represent the deviations of this point from the means Mx and My , then the slope of this line is $\frac{y}{x}$ or is the deviation of the point from the mean of the Y's divided by the deviation of the point from the mean of the X's. Since the slope is always the same and since the line passes through O_1, we may consider this the origin and write the equation of CC': $y = mx$. Now if we find by measurement the x and y value of any one point, we may find

m and thus write the equation of the line CC'. The difficulty here is that y is measured in inches and x is measured in pounds. This may be cared for by dividing each by its standard deviation which measures the variation of each series about its mean. Therefore, if we consider the ratio $\frac{y}{\sigma_y} \div \frac{x}{\sigma_x}$ we have a measure of the degree of relationship showing the trend of the variations of the x's and y's. This ratio $\frac{\frac{y}{\sigma_y}}{\frac{x}{\sigma_x}}$ we define as r, the coefficient of correlation.

Therefore, we write $\frac{y}{\sigma_y} = r \frac{x}{\sigma_x}$ as the equation of the line CC'. Similarly if we found the line best fitting the means of the rows and called it RR', we could show the equation of RR' to be $\frac{x}{\sigma_x} = r \frac{y}{\sigma_y}$ in the same way.

Having explained the meaning of correlation in terms of these equations, we shall now attempt to develop them in a more rigorous manner.

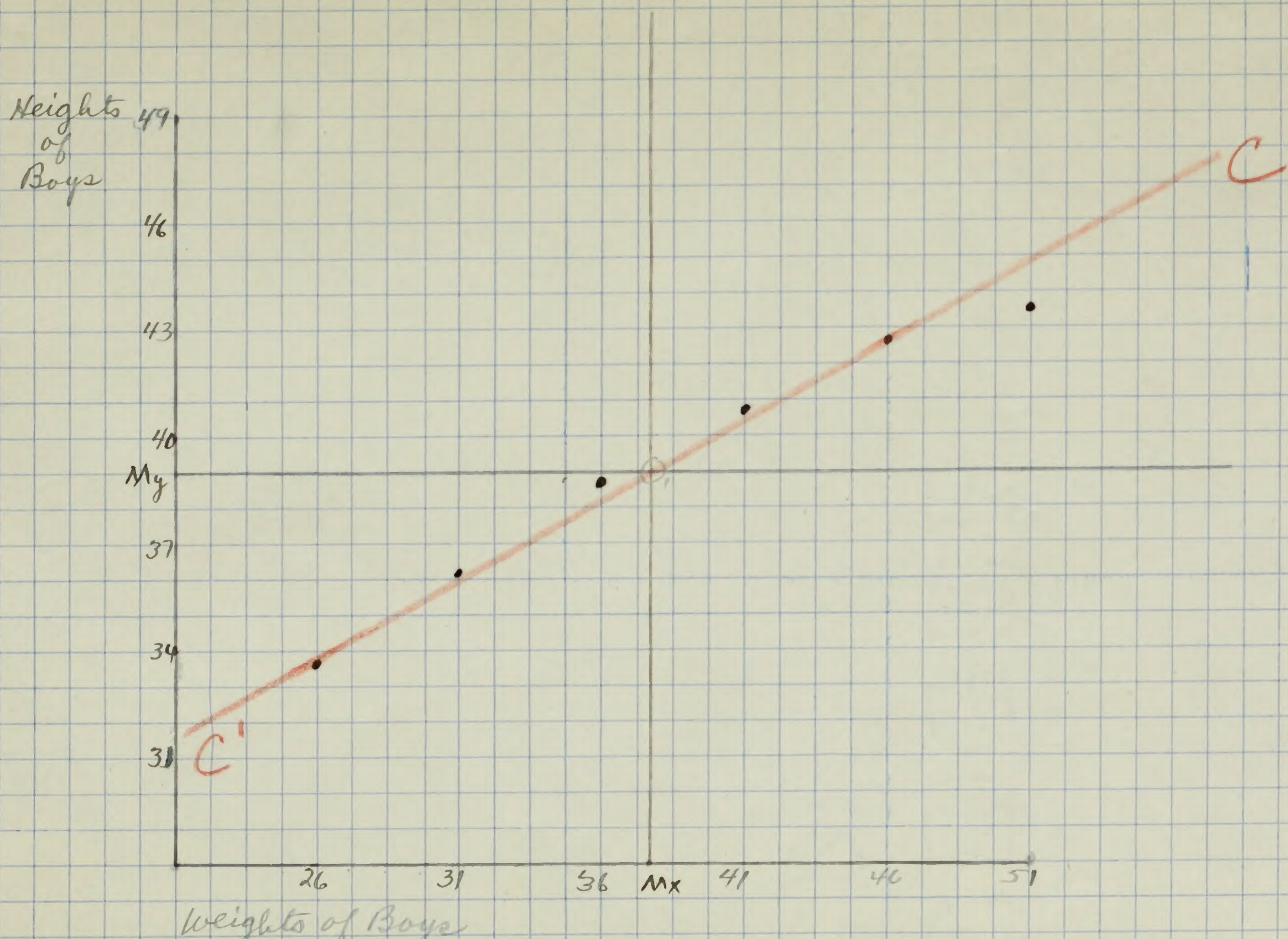
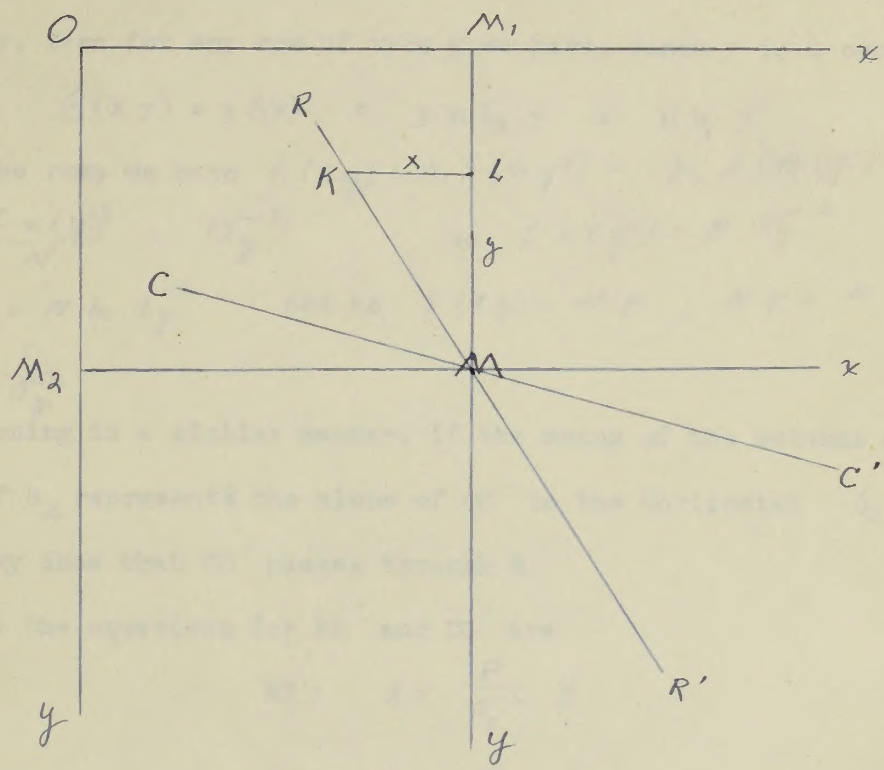


Fig. II. Graph Showing Estimated Line of Best Fit For the Means of the Columns.
CC' is the Regression Line of y on x

Chapter III Development of the Regression Lines.

A. An Indirect Geometrical Approach

"Introduction to Theory of Statistics" Yule, P. 170 ff.



Suppose we had a distribution in which all the means of the rows were on the line RR' and let x be the deviations of the rows from M_1y and y be the deviations from M_2x. Call the slope of the line RR' to My, b_1; the equation of RR' is then $x = b_1 y$. Then in any row of type y in which the number of observations is n, $\frac{\Sigma(x)}{n}$ = the deviation of the mean point of that row. Since that point is on RR', we may now rewrite the equation of RR':

$\frac{\Sigma(x)}{n} = b_1 y$ or $\Sigma(x) = n b_1 y$

If we consider this for the entire distribution, we write $\Sigma x = b_1 \Sigma(ny)$ where $\Sigma(x)$ is the sum of the deviations of all the X's from My and $\Sigma(ny)$ is the sum of the deviations of all the $\bar{Y}$'s from M_2x. But $\Sigma(ny)=0$ because M_2x is the mean of all the $\bar{Y}$'s. $\therefore$ $\Sigma(x) = b_1 \Sigma(ny) = 0$

Since $\Sigma(x) = 0$, the sum of the deviations of all the $\bar{X}$'s from My is zero, so My must be the mean of all the $\bar{X}$'s and must cut OX at M_1, the mean of $\bar{X}$. In this way we have shown M to be the mean of the entire distribution.

Now RR' passes through M, a point which we may locate for any distribution. Therefore, to write the equation for the line we have only to determine b_1, its slope.

If we define $p = \frac{1}{N}\Sigma(x,y)$ the mean product of all associated deviations of x and y, then for any row of type y we have, since y is a constant

$$\Sigma(x\,y) = y\,\Sigma(x) \quad = \quad y\,n\,b_1\,y \quad = \quad n\,b_1\,y^2$$

For all the rows we have $\Sigma(x\,y) = b_1\Sigma(n\,y^2) = b_1\,\Sigma(n)(y^2)$

but $\frac{\Sigma n(y^2)}{N} = \sigma_y^2$ so $\Sigma n(y^2) = N\sigma_y^2$

$\therefore \Sigma(x\,y) = N\,b_1\,\sigma_y^2$ and as $\Sigma(x\,y) = N\,p$, $N\,p = N\,b_1\,\sigma_y^2$

and $b_1 = \frac{p}{\sigma_y^2}$

Reasoning in a similar manner, if the means of the columns all lie on CC' and if b_2 represents the slope of CC' to the horizontal $b_2 = \frac{p}{\sigma_x^2}$

Also we may show that CC' passes through M.

Hence the equations for RR' and CC' are

$$RR': \quad x = \frac{p}{\sigma_x^2}\,y$$

$$CC': \quad y = \frac{p}{\sigma_y^2}\,x$$

The forms of these equations are not suitable for calculation, so we must rewrite them. If we set $r = \frac{p}{\sigma_x\sigma_y}$ we introduce the usual notation for the coefficient of correlation and write:

$$RR': \quad x = r\,\frac{\sigma_x}{\sigma_y}\,y$$

$$CC': \quad y = r\,\frac{\sigma_y}{\sigma_x}\,x$$

These are the same equations we arrived at in our general discussion above.

In this discussion we assumed that the means of the rows would fall on RR' and the means of the columns would fall on CC'. We must now consider the more usual situation where this does not occur.

If the values of x and y (the deviations from M_y and M_x) be found for all associated pairs of values, then we find:

A. (1) $\sum (x-b_1 y)^2 = N\sigma_x^2(1-r^2)$ when $b_1 = r\dfrac{\sigma_x}{\sigma_y}$

and where x is the actual deviation from the mean and $b_1 y$ is the estimated deviation.

Proof.

(1) $(x-b_1 y)^2 = x^2 - 2xb_1 y + b_1^2 y^2$

(2) $\sum (x-b_1 y)^2 = \sum x^2 - 2b_1 \sum xy + b_1^2 \sum y^2$

(3) $= N\sigma_x^2 - 2b_1 NP + b_1^2 N\sigma_y^2$

(4) $= N\sigma_x^2 - 2b_1 N b_1 \sigma_y^2 + b_1^2 N\sigma_y^2$

(5) $= N\sigma_x^2 - N\sigma_y^2 b_1^2$

(6) $= N\sigma_x^2 - N\sigma_y^2 \cdot r^2 \dfrac{\sigma_x^2}{\sigma_y^2}$

(7) $\sum (x-b_1 y)^2 = N\sigma_x^2(1-r^2)$

Q.E.D.

Now if b_1 equals any other value such as $b_1 = (r+\delta)\dfrac{\sigma_x}{\sigma_y}$, then

B. $\sum (x-b_1 y)^2 = N\sigma_x^2(1-r^2+\delta^2)$

Proof.

(1) $\sum (x-b_1 y)^2 = \sum \left[x-(r+\delta)\dfrac{\sigma_x}{\sigma_y}y\right]^2$

(2) $= \sum \left[x^2 - 2xy(r+\delta)\dfrac{\sigma_x}{\sigma_y} + (r+\delta)^2\dfrac{\sigma_x^2}{\sigma_y^2}y^2\right]$

(3) $= \sum\left[x^2 - 2xyr\dfrac{\sigma_x}{\sigma_y} + r^2\dfrac{\sigma_x^2}{\sigma_y^2}y^2\right] + \sum\left[-2xy\delta\dfrac{\sigma_x}{\sigma_y} + 2r\delta\dfrac{\sigma_x^2}{\sigma_y^2}y^2 + \delta^2\dfrac{\sigma_x^2}{\sigma_y^2}y^2\right]$

(4) $= \sum\left(x - r\dfrac{\sigma_x}{\sigma_y}y\right)^2 - 2\delta\dfrac{\sigma_x}{\sigma_y}\sum(xy) + 2r\delta\dfrac{\sigma_x^2}{\sigma_y^2}\sum(y^2) + \delta^2\dfrac{\sigma_x^2}{\sigma_y^2}\sum(y^2)$

(5) $= \sum\left(x - r\dfrac{\sigma_x}{\sigma_y}y\right)^2 - 2\delta\dfrac{\sigma_x}{\sigma_y}\cdot Nr\sigma_x\sigma_y + 2r\delta\dfrac{\sigma_x^2}{\sigma_y^2}\cdot N\sigma_y^2 + \delta^2\dfrac{\sigma_x^2}{\sigma_y^2}\cdot N\sigma_y^2$

(6) $= \sum\left(x - r\dfrac{\sigma_x}{\sigma_y}y\right)^2 - 2Nr\delta\sigma_x^2 + 2Nr\delta\sigma_x^2 + N\delta^2\sigma_x^2$

(7) $= \sum\left(x - r\dfrac{\sigma_x}{\sigma_y}y\right)^2 + N\delta^2\sigma_x^2$

(8) $\sum (x-b_1 y)^2 = N\sigma_x^2(1-r^2) + N\delta^2\sigma_x^2 = N\sigma_x^2(1-r^2+\delta^2)$

Now we showed when $b_1 = r\dfrac{\sigma_x}{\sigma_y}$

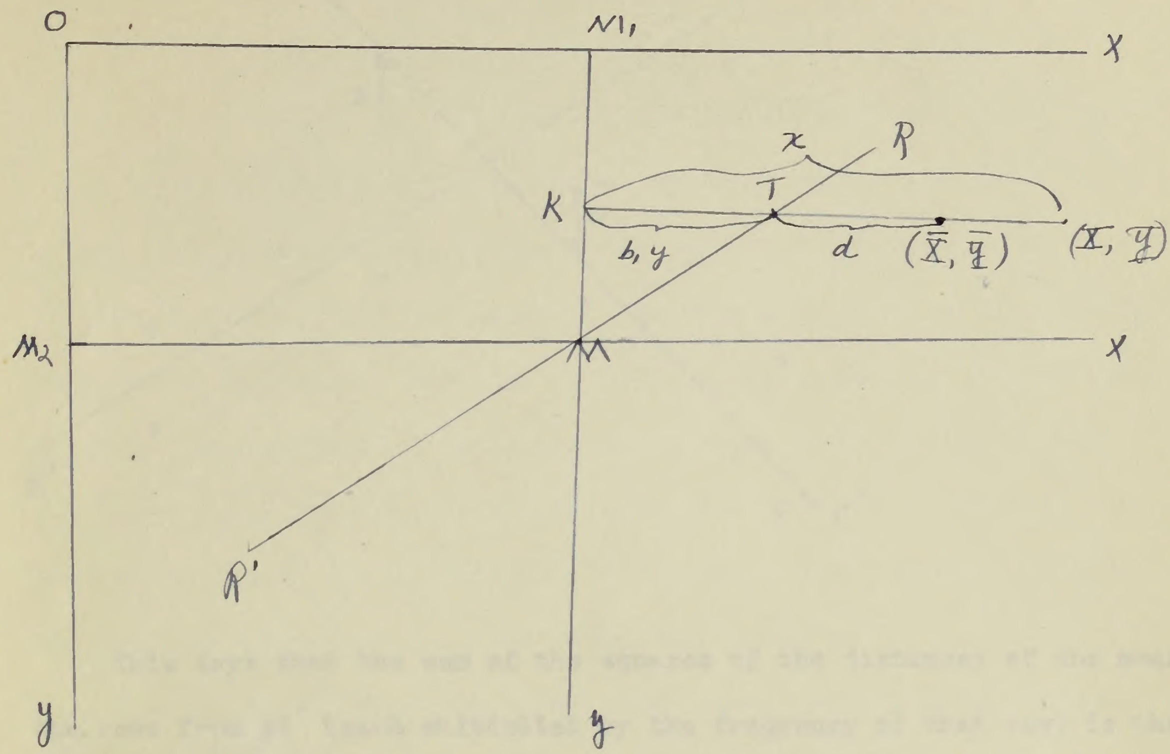
A. $\sum (x-b_1 y)^2 = N\sigma_x^2(1-r^2)$

and when $b_1 = (r+\delta)\dfrac{\sigma_x}{\sigma_y}$

B. $\sum (x-b_1 y)^2 = N\sigma_x^2(1-r^2+\delta^2)$

The right-hand side of B is obviously greater than the right-hand side of A;

so $\sum (x - b_1 y)^2$ is a minimum when $b_1 = r \frac{\sigma_x}{\sigma_y}$.



Let us consider the distribution in a row of type y with the origin at K and find the root mean-square deviation of the row about point T (on RR$'$). See Note #1, Page 1 of Footnotes.

The root mean-square deviation of the row about T is $\frac{1}{n} \sum (x - b_1 y)^2$ where n is the frequency of the row.

$$\therefore \frac{1}{n} \sum (x - b_1 y)^2 = S^2_{ax} + d^2$$

where S^2_{ax} is the standard deviation of the row.

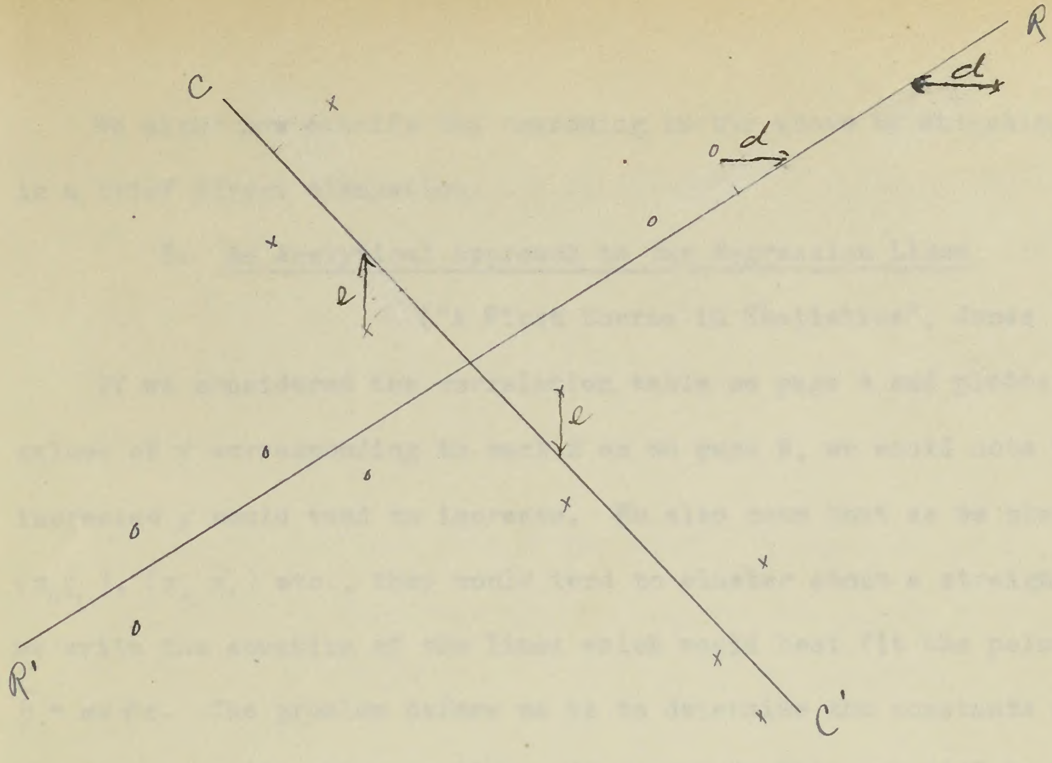
So for the row $\sum (x - b_1 y)^2 = n S^2_{ax} + n d^2$

For the entire distribution then

$$\sum (x - b_1 y)^2 = \sum (n S^2_{ax}) + \sum (n d^2)$$

$\sum (n S^2_{ax})$ is the sum of the standard deviations of the rows and remains unchanged regardless of the slope of RR$'$, so d and $(x - b_1 y)$ are the only terms affected by RR$'$.

Now if $\sum (x - b_1 y)^2$ is a minimum, so $\sum (n d^2)$ must be a minimum for the value of $b_1 = r \frac{\sigma_x}{\sigma_y}$.



This says that the sum of the squares of the distances of the means of the rows from RR' (each multiplied by the frequency of that row) is the lowest possible when $b_1 = r \frac{\sigma_x}{\sigma_y}$.

The same can be proved in like manner for CC'; that is, the sum of the squares of the distances of the means of the columns from CC' is the lowest possible when $b_2 = r \frac{\sigma_y}{\sigma_x}$ and the equation of CC' is $y = b_2 x$.

Therefore, the equations

$$x = r \frac{\sigma_x}{\sigma_y} \quad \text{and} \quad y = r \frac{\sigma_y}{\sigma_x} \quad \text{may be regarded as}$$

"(a) equations for estimating each individual x from its associated y (and y from its associated x) in such a way as to make the sum of the squares of errors of estimate the least possible, or (b) equations for estimating the mean of the x's associated with a given type of y (and the mean of the y's associated with a given type of x) in such a way as to make the sum of the squares of errors of estimate the least possible when every mean is counted once for each observation on which it is based." Yule P. 72-3

These lines are called the lines of "best fit" of the actual lines of the means.

We might now clarify the reasoning in the above by attacking the problem in a brief direct discussion.

B. <u>An Analytical Approach to the Regression Lines</u>

("A First Course in Statistics", Jones P. 104 ff.)

If we considered the correlation table on page 4 and plotted the mean values of y corresponding to each x as on page 9, we would note that as x increased y would tend to increase. We also note that as we plot the points $(x_1, \bar{y}_1)$, $(x_2, \bar{y}_2)$ etc., they would tend to cluster about a straight line. If we write the equation of the lines which would best fit the points, it is $\bar{y} = mx + c$. The problem before us is to determine the constants m and c so that we may write this equation. If we can do this, we will be able to find the best average value of y corresponding to any x.

Now $\bar{y}_1$, $\bar{y}_2$, $\bar{y}_3$ etc., were the best values of y corresponding to x_1, x_2, x_3 etc., so if we rewrite the equation $y = mx + c$ we will be still estimating the best y corresponding to any given x and basing our work on all the observations since $\bar{y}_i$ is the best value of y in that particular column.

If $x = x_1$

$y = m x_1 + c$

But for any value x_1 of x there may be several values of y as seen in the correlation table on page 4; if y_1 is one of these values, the difference between it and the value given by the equation is $(mx_1 + c) - y_1$.

This difference measures the distance measured parallel to the y axis between the observed point (x_1, y_1) and the line $y = mx + c$. We now wish to find the equation of a line such that the sum of these differences for all paired values of x and y will be a minimum. Since some of these differences are positive and some negative, we will search for the equation which will make the sum of their squares a minimum. The problem then, is to find c and m which will make

$$(m x_1 + c - y_1)^2 + (m x_2 + c - y_2)^2 + \cdots + (m x_n + c - y_n)^2$$

a minimum.

If we consider this an expression in c, differentiate, and set equal to zero, we will have the value of c which makes the expression a minimum.

$$(mx_1 + c - y_1) + (mx_2 + c - y_2) + \cdots + (mx_n + c - y_n) = 0$$

$$m(x_1 + x_2 + \cdots + x_n) + nc - (y_1 + y_2 + \cdots + y_n) = 0$$

$$m(n\bar{x}) + nc - n\bar{y} = 0$$

$$m\bar{x} + c - \bar{y} = 0$$

This equation passes through the point $(\bar{x}, \bar{y})$, the mean of the entire distribution. This suggests that we transpose the origin to the point $(\bar{x}, \bar{y})$ so that $x - \bar{x} = d$ and $y - \bar{y} = k$, now in the equation $m\bar{x} + c - \bar{y} = o$, the value of $c = o$.

Now returning to $(mx_1 + c - y_1)^2 + \cdots + (mx_n + c - y_n)^2$ and differentiating with respect to m.

$$x_1(mx_1 + c - y_1) + x_2(mx_2 + c - y_2) + \cdots + x_n(mx_n + c - y_n) = 0$$

$$m(x_1^2 + x_2^2 + \cdots + x_n^2) + c(x_1 + x_2 + \cdots + x_n) - (x_1y_1 + x_2y_2 + \cdots + x_ny_n) = 0$$

Now replacing the x's and y's by their deviations from the mean (i.e. transposing the origin to $(\bar{x}, \bar{y})$.

$$m(d_1^2 + d_2^2 + \cdots + d_n^2) + 0(d_1 + d_2 + \cdots + d_n) - (d_1k_1 + d_2k_2 + \cdots + d_nk_n) = 0$$

$$m = \frac{d_1k_1 + d_2k_2 + \cdots + d_nk_n}{d_1^2 + d_2^2 + \cdots + d_n^2}$$

Now if $$P = \frac{d_1k_1 + d_2k_2 + \cdots + d_nk_n}{n}$$

and $$\sigma_x^2 = \frac{d_1^2 + d_2^2 + \cdots + d_n^2}{n}$$ as usual,

then $$m = \frac{nP}{n\sigma_x^2} = \frac{P}{\sigma_x^2}$$

Thus we have shown that if we considered the equation $y = mx + c$ and transferred the origin to $\bar{x}, \bar{y}$ this equation would be

$$k = md + c$$

or

$$y - \bar{y} = m(x - \bar{x}) + c$$

and $\quad C = 0, \quad m = \dfrac{\rho}{\sigma_x^2}$

$$\therefore (y - \bar{y}) = \frac{\rho}{\sigma_x^2}(x - \bar{x})$$

If $x - \bar{x} = 1$, then $y - \bar{y} = \dfrac{\rho}{\sigma_x^2}$, so $\dfrac{\rho}{\sigma_x^2}$ measures the change in the deviation of y corresponding to a unit change in the deviation of x.

If we repeated the entire discussion interchanging the x's and y's we would arrive in exactly the same steps at the result

$$(x - \bar{x}) = \frac{\rho}{\sigma_y^2}(y - \bar{y})$$

Thus if $(y - \bar{y}) = 1$, $(x - \bar{x}) = \dfrac{\rho}{\sigma_y^2}$, so $\dfrac{\rho}{\sigma_y^2}$ measures the deviation in x from the mean of x corresponding to a unit deviation in y from the mean of y.

Therefore, either $\dfrac{\rho}{\sigma_x^2}$ or $\dfrac{\rho}{\sigma_y^2}$ may be considered as good measures for the correlation between x and y; they are not alike because $\dfrac{\rho}{\sigma_x^2}$ gives the change in y corresponding to a unit change in x and $\dfrac{\rho}{\sigma_y^2}$ gives the change in x corresponding to a unit change in y. If we wish to compare these changes, we must reduce them to ratios which will be comparable, so we divide $x - \bar{x}$ by the standard deviation σ_x and $y - \bar{y}$ by σ_y and compare

$$y - \bar{y} = \frac{\rho}{\sigma_x^2}(x - \bar{x})$$

Divide both sides by σ_y

$$\frac{y - \bar{y}}{\sigma_y} = \frac{\rho}{\sigma_x \sigma_y}\left(\frac{x - \bar{x}}{\sigma_x}\right)$$

Similarly for

$$x - \bar{x} = \frac{\rho}{\sigma_y^2}(y - \bar{y})$$

Divide by σ_x

$$\frac{x - \bar{x}}{\sigma_x} = \frac{\rho}{\sigma_x \sigma_y}\left(\frac{y - \bar{y}}{\sigma_y}\right)$$

Now we have $\dfrac{\rho}{\sigma_x \sigma_y}$ as the measure of correlation and write $r = \dfrac{\rho}{\sigma_x \sigma_y}$

Now substituting r in our equations, we have

$$y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$$

$$x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y})$$

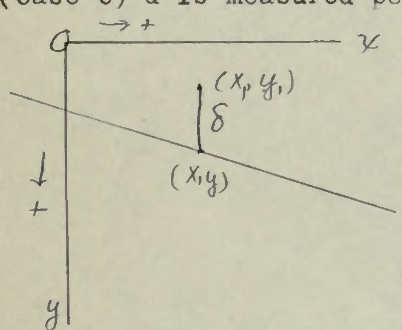
which are the equations of the lines of regression of y on x and x on y

respectively.

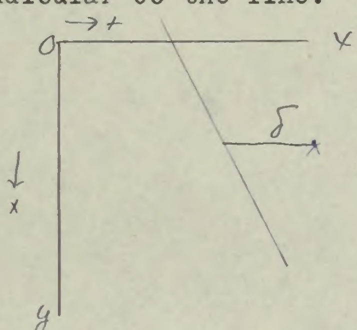
C. <u>A Development of the Regression Lines Introduced to Show the Range of Values of r. An Exact Definition of the Correlation Coefficient.</u>
("Mathematical Part of Elementary Statistics" Camp)

If we think of the general case of correlation, we could think of the data represented by dots spread over the paper. We wish to find the equation of that straight line which, on the whole, will come nearest to all these dots. That is, if we let δ be the distance between a dot and the line, we wish to make $\Sigma\,\delta^2$ a minimum.

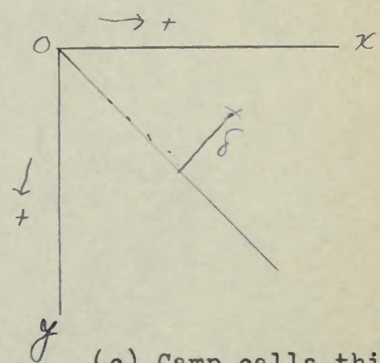
There are three cases, depending on whether (case a) d is measured parallel to the y axis; (case b) d is measured parallel to the x axis; (case c) d is measured perpendicular to the line.



(a) The regression line of y on x.



(b) The regression line of x on y.



(c) Camp calls this the "geometrically best fitting line."

A Method of finding Lines (a) and (c) by Means of Least Squares - Introduced to Show the Range of r.

<u>Case (a)</u> To obtain the equation of the regression of Y on X.

Here we wish $\Sigma\,\delta^2 f$ to be a minimum, where f represents the frequency.

(1) $\delta = y - y_1$

(2) Let $y = A + Bx_1$

(3) Then $\delta = A + Bx_1 - y_1$

(4) $\delta^2 = A^2 + B^2x_1^2 + y_1^2 + 2ABx_1 - 2Ay_1 - 2Bx_1y_1$

(5) $\frac{1}{N}\Sigma\delta^2 f = \frac{1}{N}\Sigma\left(A^2 + B^2x^2 + y^2 + 2ABx - 2Ay - 2Bxy\right) f(x,y)$

(6) $\frac{1}{N}\Sigma\delta^2 f = \frac{A^2}{N}\Sigma f(x,y) + \frac{B^2}{N}\Sigma x^2 f(x,y) + \frac{1}{N}\Sigma y^2 f(x,y) + 2\frac{AB}{N}\Sigma x f(x,y)$
$- 2\frac{A}{N}\Sigma y f(x,y) - \frac{2B}{N}\Sigma xy\, f(x,y)$

(7) $\frac{1}{N}\Sigma\delta^2 f = A^2 + B^2\sigma_x^2 + \sigma_y^2 - 2Br\sigma_x\sigma_y$

(8) The expression on the left of (7) is the sum of squares and is positive, so the expression on the right is positive. So if we take A = 0, we can then find what value of B will make this expression a minimum. That is, we wish to make

(9) $\sigma_y^2 + \sigma_x^2\left(B^2 - 2Br\frac{\sigma_y}{\sigma_x}\right)$ a minimum.

(10) This expression will be a minimum when $B^2 - 2Br\frac{\sigma_y}{\sigma_x}$ is a minimum.

(11) Differentiating with respect to B and equaling to zero, we get

$$2B - 2r\frac{\sigma_y}{\sigma_x} = 0$$

(12) $B = \frac{r\sigma_y}{\sigma_x}$

(13) Substituting (12) in (2) and noting that A = 0 we get

$$y = \frac{r\sigma_y}{\sigma_x}x$$

This is the equation for the regression of $\underline{Y}$ on $\underline{X}$

<u>Case (b)</u>

In the same manner interchanging the y's and x's we get the equation of the regression of $\underline{X}$ on $\underline{Y}$

$$x = \frac{r\sigma_x}{\sigma_y}y$$

(14) Now in (7) we had

$$\frac{1}{N}\Sigma\delta^2 f = \sigma_y^2 + A^2 + B^2\sigma_x^2 - 2Br\sigma_y\sigma_x$$

but A = 0 and B = r $\frac{\sigma_y}{\sigma_x}$.

(15) Hence $\frac{1}{N}\Sigma\delta^2 f = \sigma_y^2 + r^2\frac{\sigma_y^2}{\sigma_x^2}\sigma_x^2 - 2r^2\frac{\sigma_y}{\sigma_x^2}\sigma_y\sigma_x^2$

(16) $\frac{1}{N}\Sigma\delta^2 f = \sigma_y^2(1-r^2)$

Now the left side of this equation is positive, since it is the sum of squares, so the right side is positive. Hence

$$1 - r^2 \geqq 0$$

and

$$-1 \leqq r \leqq 1$$

Also in similar manner in the case of the equation for the regression of X on Y we may show

$$\frac{1}{N}\sum \delta^2 f = \sigma_x^2(1-r^2)$$

and $-1 \leq r \leq 1$

Now to return to Case (c) and to find the equation of the "Geometrically best-fitting line."

(1) In Analytic Geometry, the formula for the distance from the point (x_1, y_1) to the line $Ax + By + C = 0$ is

$$d = \frac{Ax_1 + By_1 + C}{\pm\sqrt{A^2+B^2}}$$ where d is positive.

(2) Our equation is

$y = A + Bx$ or $Bx - y + A = 0$ and

the point is (x_1, y_1)

(3) Hence $\delta = \dfrac{Bx_1 - y_1 + A}{\pm\sqrt{B^2+1}}$

(4) $\frac{1}{N}\sum \delta^2 f = \frac{1}{N}\sum \left(\frac{Bx - y + A}{\pm\sqrt{B^2+1}}\right)^2 f(x,y)$

(5) $\frac{1}{N}\sum \delta^2 f = \frac{1}{B^2+1}\left[B^2\sigma_x^2 + \sigma_y^2 + A^2 - 2B\,P_{xy}\right]$

(6) $\frac{1}{N}\sum \delta^2 f = \frac{1}{B^2+1}\left[B^2\sigma_x^2 + \sigma_y^2 + A^2 - 2Br\sigma_x\sigma_y\right]$

(7) Here again, the left side is positive, so the right side is positive.

If we let A = 0, we may solve for that value of B which will make

$$\frac{B^2\sigma_x^2 + \sigma_y^2 - 2Br\sigma_x\sigma_y}{B^2+1}$$ a minimum.

(8) In this problem, however, we are interested only when the standard deviation is used as a unit, so we first set $\sigma_x = \sigma_y = 1$

(9) Therefore (7) becomes

$$\frac{B^2 + 1 - 2Br}{1 + B^2}$$

and we wish to find the value of B which will make this a minimum.

(10) Rewrite (9)

$$1 - \frac{2Br}{B^2+1}$$

(11) Differentiating with respect to B and equating to zero

$$\beta = \pm 1$$

(12) When $r > 0$

$$1 - \frac{2\beta r}{1+\beta^2} \text{ will be a minimum when } \beta = +1$$

When $r < 0$

$$1 - \frac{2\beta r}{1+\beta^2} \text{ will be a minimum when } \beta = -1$$

When $r = 0$

$$1 - \frac{2\beta r}{1+\beta^2} = 1 \text{ and cannot be a minimum.}$$

(13) So we may write the equation for the "geometrically best-fitting line"

$y = x$ if $r > 0$ and $\sigma_x = \sigma_y = 1$

$y = -x$ if $r < 0$ and $\sigma_x = \sigma_y = 1$

Now if we let $\sigma_x = \sigma_y = 1$ and rewrite the equations for the regression lines, we have

(a) The equation of the regression of Y on X

$$y = rx$$

(b) The equation of the regression of X on Y

$$x = ry$$

(c) The equation of the "geometrically best-fitting" line

$y = x$ if $r > 0$

$y = -x$ if $r < 0$

(d) When δ is measured parallel to the y axis $\frac{1}{N}\sum \delta^2 f = 1 - r^2$

(e) When δ is measured parallel to the x axis $\frac{1}{N}\sum \delta^2 f = 1 - r^2$

(f) When δ is the perpendicular distance from the point to the line

$$\frac{1}{N}\sum \delta^2 f = \frac{1}{B^2+1}\,(1+B^2-2Br)$$

when $r>0 \quad B=1$

$r<0 \quad B=-1$

so $$\frac{1}{N}\sum\delta^2 f = \frac{2-2r}{2} = 1-r \qquad r>0$$

$$\frac{1}{N}\sum\delta^2 f = \frac{2+2r}{2} = 1+r \qquad r<0$$

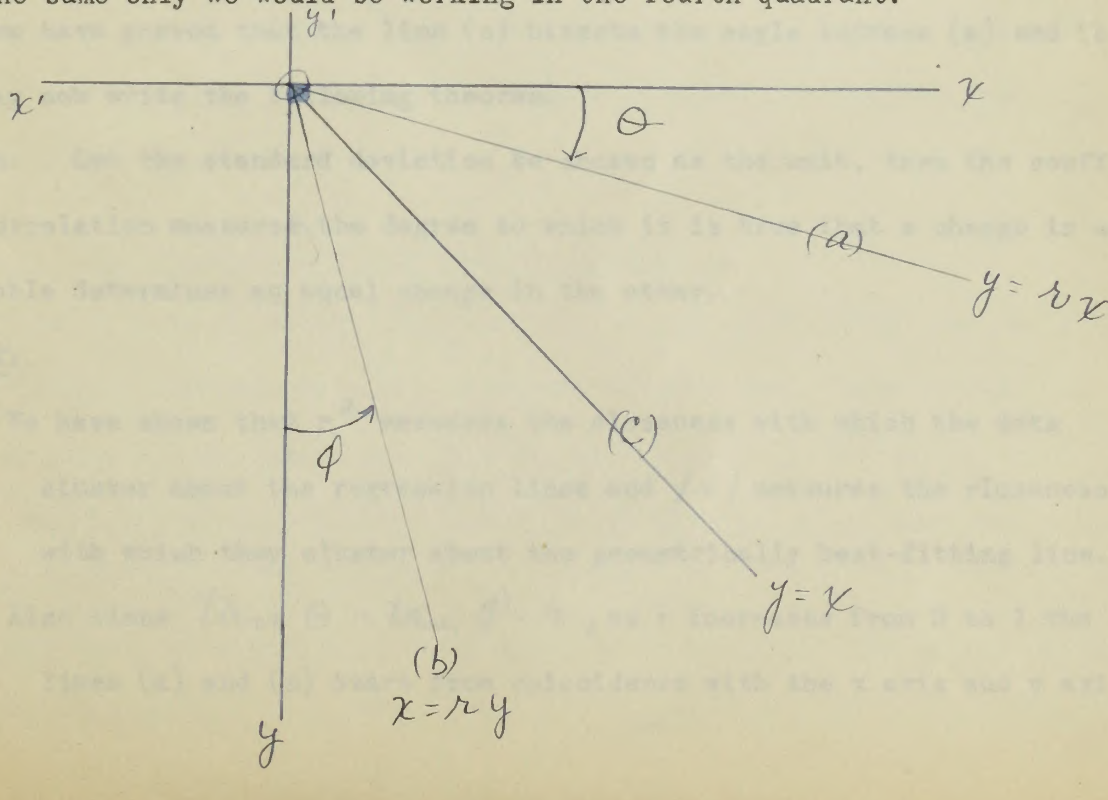
Therefore, from (d), (e), and (f) we may write

Th. $|r|$ measures the closeness with which the dots cluster about the geometrically best-fitting line; r^2 measures the closeness with which the dots cluster about the regression lines (distances in the last case being measured parallel to the y and x axis).

Now we are in a position to prove the following theorem.

Th. If $\sigma_x = \sigma_y = 1$, the line y = x, when r > o, bisects the angle between the lines $y = rx$ and $x = ry$; when $r<0$, the line $y=-x$ bisects the angle between the lines $y = rx$ and $x = ry$

We shall prove the first part, for the proof of the second part would be the same only we would be working in the fourth quadrant.



Proof:

(1) Since there is no constant term, these lines all pass through the origin.
(Note the positive direction of y is downward)

(2) The equation for the line (a) is

$$y = r x$$

so $r = \frac{y}{x}$

$\therefore r = \tan \theta$

(3) Similarly for (b) $r = \frac{x}{y}$

so $r = \tan \phi$

(4) $\therefore$ from (2) and (3) $\tan \phi = \tan \theta$

$\therefore \phi = \theta$

(5) The line y = x bisects the angle x o y

(6) So the angle between (a) and (c) is

$$45^\circ - \theta$$

The angle between (b) and (c) is

$$45^\circ - \phi$$

(7) $\therefore$ Since $\phi = \theta$

$$45^\circ - \phi = 45^\circ - \theta$$

And we have proved that the line (c) bisects the angle between (a) and (b).

We may now write the following theorem.

Th. Let the standard deviation be chosen as the unit, then the coefficient of correlation measures the degree to which it is true that a change in one variable determines an equal change in the other.

Proof:

(1) We have shown that r^2 measures the closeness with which the dots cluster about the regression lines and $|r|$ measures the closeness with which they cluster about the geometrically best-fitting line.

(2) Also since $\tan \theta = \tan \phi = r$, as r increases from 0 to 1 the lines (a) and (b) start from coincidence with the x axis and y axis,

respectively, and rotate with equal angular velocities in the direction of c. When r = 1, they coincide with c.

(Note when r = -1, these lines coincide with (y = -x)

(3) For points on c a change in one variable determines an equal change in the other.

For the slope of the line is 1 and if x'', y'' and x', y' are two points on it

$$\frac{y''-y'}{x''-x'} = 1$$

$$y''-y' = x''-x'$$

(4) Therefore, the larger r is, the nearer (a) and (b) come to coincidence with (c); the nearer the dots lie to c; and the nearer we have the condition that an equal change in one variable produces an equal change in the other.

We might now sum this discussion with the following statement of the above theorem.

Th. The coefficient of correlation measures the degree to which it is true that a relative change in one variable determines an equal relative change in the other. By a relative change is meant the ratio of the absolute change to the standard deviation.

D. <u>The Standard Deviation of the Arrays</u>

When we have found the equations of the regression lines, we are interested in knowing the dispersion of the rows and columns about these lines.

On page 11, we found that $\sum (x - b_1 y)^2 = N\sigma_x^2(1-r^2)$

so we might define the standard deviation of the rows about the regression line of x on y as $\sigma_x\sqrt{1-r^2}$

Similarly we might define the standard deviation of the columns about the regression line of y on x as $\sigma_y\sqrt{1-r^2}$

Chapter IV The Correlation Surface

(Elderton - "Frequency, Curves and Correlation")
(Forsyth - "Mathematical Analysis of Statistics")

The equation representing the correlation surface where the probability of the joint occurrence of deviations of x and y from their respective means, whether they are dependent or independent, can be written

$$z = k\,e^{-\frac{1}{2}(ax^2 + 2hxy + by^2)}$$

This equation is developed by Elderton in his "Frequency, Curves and Correlation" and is assumed in this discussion.

Our interest now is to replace the constants in the equation by expressions which will be of service in interpreting correlation tables; i.e. the standard deviations of x and y and the coefficient of correlation. This can be done by finding the volume N under the surface z.

$$\therefore N = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z\,dx\,dy$$

Let us consider first $\int_{-\infty}^{\infty} z\,dx$

(1) $= \int_{-\infty}^{\infty} k\,e^{-\frac{1}{2}(ax^2+2hxy+by^2)}\,dx$

(2) $= k\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(ax^2+2hxy+\frac{h^2y^2}{a}-\frac{h^2y^2}{a}+by^2\right)}\,dx$

(3) $= k\,e^{-\frac{a}{2}\left(\frac{by^2}{a}-\frac{h^2y^2}{a^2}\right)}\int_{-\infty}^{\infty} e^{-\frac{a}{2}\left(x+\frac{hy}{a}\right)^2}\,dx$

(4) Now evaluate $\int_{-\infty}^{\infty} e^{-\frac{a}{2}\left(x+\frac{hy}{a}\right)^2}\,dx$

Let $X = x + \frac{hy}{a}$ $\qquad dX = dx$

$$\int_{0}^{\infty} e^{-\frac{a}{2}\left(x+\frac{hy}{a}\right)^2}dx = \frac{1}{2}\sqrt{\frac{2\pi}{a}}$$

Since we will show later that any cross section of the normal surface made by a plane parallel to the yz plane or parallel to the xz plane is a normal curve and is symmetrical, $2\int_{0}^{\infty} e^{-\frac{a}{2}X^2}\,dX = \int_{-\infty}^{\infty} e^{-\frac{a}{2}X^2}\,dX$

(5) $\therefore \int_{-\infty}^{\infty} e^{-\frac{a}{2}\left(x+\frac{hy}{a}\right)^2}\,dx = \sqrt{\frac{2\pi}{a}}$

(6) So $\int_{-\infty}^{\infty} z\, dx = k\sqrt{\frac{2\pi}{a}}\, e^{-\frac{y^2}{2}\left(b - \frac{h^2}{a}\right)}$

Since we integrated $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z\, dx\, dy$ with respect to x first, we have then in $k\sqrt{\frac{2\pi}{a}}\, e^{-\frac{y^2}{2}\left(b - \frac{h^2}{a}\right)}$ a typical y section; so we may write

$$z = k\sqrt{\frac{2\pi}{a}}\, e^{-\frac{y^2}{2}\left(b - \frac{h^2}{a}\right)}$$

where z is the frequency of this particular section.

Since it may be shown that a normal curve can be written $z = m\, e^{-\frac{y^2}{2}\cdot\frac{1}{\sigma_y^2}}$ we have

(7) $\frac{1}{\sigma_y^2} = b\left(1 - \frac{h^2}{ab}\right)$

(8) Now if we integrate $\int_{-\infty}^{\infty} z\, dy$ in exactly the same manner we find

$$\frac{1}{\sigma_x^2} = a\left(1 - \frac{h^2}{ab}\right)$$

(9) Let $r = -\frac{h}{\sqrt{ab}}$, $r^2 = \frac{h^2}{ab}$

(10) From (7) $\frac{1}{\sigma_y^2} = b(1-r^2)$, $b = \frac{1}{\sigma_y^2(1-r^2)}$

(11) From (8) $\frac{1}{\sigma_x^2} = a(1-r^2)$, $a = \frac{1}{\sigma_x^2(1-r^2)}$

(12) In (9) $h = -r\sqrt{ab}$

so $h = \frac{-r}{\sigma_x \sigma_y (1-r^2)}$

We now have the constants a, b, and h expressed in the desired terms; only k remains.

Returning to

$N = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z\, dx\, dy$ and substituting for $\int_{-\infty}^{\infty} z\, dx$ its value $k\sqrt{\frac{2\pi}{a}}\, e^{-\frac{y^2}{2\sigma_y^2}}$, we have

(13) $N = k\sqrt{\frac{2\pi}{a}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma_y^2}}\, dy$

(14) $\int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma_y^2}}\, dy = \sigma_y\sqrt{2\pi}$

(15) $\therefore N = k\sqrt{\frac{2\pi}{a}}\cdot \sigma_y\sqrt{2\pi} = \frac{2\pi k \sigma_y}{\sqrt{a}}$

(16) In (11) $a = \frac{1}{\sigma_x^2(1-r^2)}$

So $N = \frac{2\pi k \sigma_y}{\sigma_x \sqrt{1-r^2}} = 2\pi k \sigma_x \sigma_y \sqrt{1-r^2}$

(17) $k = \frac{N}{2\pi \sigma_x \sigma_y \sqrt{1-r^2}}$

(18) Now return to the equation

$$z = k e^{-\frac{1}{2}(ax^2 + 2hxy + by^2)}$$

and replace a, h, k, b by their respective values we have

$$z = \frac{N}{2\pi \sigma_x \sigma_y \sqrt{1-r^2}} e^{-\frac{1}{2(1-r^2)}\left(\frac{x^2}{\sigma_x^2} - \frac{2xyr}{\sigma_x \sigma_y} + \frac{y^2}{\sigma_y^2}\right)}$$

This is the equation for the correlation surface and r is the coefficient of correlation.

The equation for the normal curve with the x axis as its mean may be shown to be

$$\phi(y) = \frac{N}{\sigma_y \sqrt{2\pi}} e^{-\frac{y^2}{2\sigma_y^2}}$$

and that with the y axis as its mean

$$\phi(x) = \frac{N}{\sigma_x \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_x^2}}$$

so the probability of the deviation of any y from its mean is

$$\frac{N}{\sigma_y \sqrt{2\pi}} e^{-\frac{y^2}{2\sigma_y^2}}$$

and the probability of the deviation of any x from its mean is

$$\frac{N}{\sigma_x \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_x^2}}$$

Hence, if these probabilities are independent, the probability of the joint occurrence of these deviations is

(19) $$z_i = \frac{N^2}{\sigma_x \sigma_y 2\pi} e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right)}$$

which might be considered the equation of a surface.

Now in (18) we showed

$$z = \frac{N}{2\pi \sigma_x \sigma_y \sqrt{1-r^2}} e^{-\frac{1}{2(1-r^2)}\left(\frac{x^2}{\sigma_x^2} - \frac{2xyr}{\sigma_x \sigma_y} + \frac{y^2}{\sigma_y^2}\right)}$$

was the probability of the joint occurrence of deviations of x and y, whether dependent or independent.

If we consider the case where $N = 1$ and $r = 0$, then $z = z_i$ and we see that when $r = 0$, $z = z_i$. That is when $r = 0, z$ is the formula for the probabilities when they are independent. This would suggest that we consider r a measure for the dependence or correlation of the variables x and y.

Chapter V The Product-Moment Formula for Correlation.

(Jones - "A First Course in Statistics" P. 278)

Since $z = \frac{N}{2\pi\sigma_x\sigma_y\sqrt{1-r^2}} e^{-\frac{1}{2(1-r^2)}\left(\frac{x^2}{\sigma_x^2} - \frac{2rxy}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2}\right)}$, and if we suppose we have n pairs of associated values of x and y

(x_1, y_1) (x_2, y_2) - - - - - - - - - - (X_n, Y_n)

then any x_1 would occur with its y_1 in the relation

$$z = \frac{N}{2\pi\sigma_x\sigma_y\sqrt{1-r^2}} e^{-\frac{1}{2(1-r^2)}\left(\frac{x_1^2}{\sigma_x^2} - \frac{2rx_1y_1}{\sigma_x\sigma_y} + \frac{y_1^2}{\sigma_y^2}\right)}$$

The probability $\phi(x,y)$ that each x would occur with its associated y, assuming the associated pairs (x_1, y_1), (x_2, y_2) - - - - (x_n, y_n) were observed independently would be

$$\phi(x,y) = \frac{N}{2\pi\sigma_x\sigma_y\left(\sqrt{1-r^2}\right)^n}\left\{e^{-\frac{1}{2(1-r^2)}\left(\frac{x_1^2}{\sigma_x^2} - \frac{2rx_1y_1}{\sigma_x\sigma_y} + \frac{y_1^2}{\sigma_y^2}\right)} \cdots\cdots e^{-\frac{1}{2(1-r^2)}\left(\frac{x_n^2}{\sigma_x^2} - \frac{2rx_ny_n}{\sigma_x\sigma_y} + \frac{y_n^2}{\sigma_y^2}\right)}\right\}$$

$$= \frac{N}{2\pi\sigma_x\sigma_y}\cdot\left\{\frac{1}{(1-r^2)^{\frac{n}{2}}}\cdot e^{-\frac{1}{2(1-r^2)}\left[\frac{\Sigma x^2}{\sigma_x^2} - \frac{2r\Sigma(xy)}{\sigma_x\sigma_y} + \frac{\Sigma y^2}{\sigma_y^2}\right]}\right\}$$

Let $K = \frac{\Sigma(xy)}{N\sigma_x\sigma_y}$ and substitute

$$\phi(x,y) = \frac{N}{2\pi\sigma_x\sigma_y}\left\{(1-r^2)^{-\frac{n}{2}}\cdot e^{-\frac{1}{2(1-r^2)}\left[\frac{\Sigma(x^2)}{\sigma_x^2} + \frac{\Sigma(y^2)}{\sigma_y^2} - 2rnk\right]}\right\}$$

But

$$\frac{\Sigma(x^2)}{\sigma_x^2} = \frac{\Sigma(y^2)}{\sigma_y^2} = n$$

$$\phi(x,y) = \frac{N}{2\pi\sigma_x\sigma_y}\left\{(1-r^2)^{-\frac{n}{2}}\cdot e^{-\frac{1}{2(1-r^2)}\left[2n - 2rnK\right]}\right\}$$

$$\phi(x,y) = \frac{n}{2\pi\sigma_x\sigma_y}\left\{(1-r^2)^{-\frac{n}{2}}\cdot e^{-\frac{n}{(1-r^2)}[1-rk]}\right\}$$

But $(1-r^2)^{-\frac{n}{2}} = e^{-\frac{n}{2}\log(1-r^2)}$

$$\therefore f(x,y) = \frac{n}{2\pi\sigma_x\sigma_y} e^{-\frac{n}{2}\log(1-r^2)} \cdot e^{-\frac{n}{1-r^2}(1-rk)}$$

$$f(x,y) = \frac{n}{2\pi\sigma_x\sigma_y} \cdot \frac{1}{e^{\frac{n}{2}\log(1-r^2)+\frac{n}{1-r^2}(1-rk)}}$$

This probability would be the greatest when

$\frac{1}{2}\log(1-r^2) + \frac{1}{1-r^2}(1-rk)$ is the least.

Differentiate with respect to r and equate to 0. This gives the value of r which will make the expression a minimum and will make the probability the greatest.

$$\frac{1}{2} \cdot \frac{-2r}{1-r^2} + \frac{(1-r^2)(-k)+(1-rk)(2r)}{(1-r^2)^2} = 0$$

$$-r + r^3 + 2r - k - kr^2 = 0$$

$$(r^2+1)(r-k) = 0$$

$$r = k$$

The first derivative is $r^3 - kr^2 + r - k$

The second is $3r^2 - 2rk + 1$ and when $r = k$, this is $r^2 + 1$.

Therefore the above expression is a minimum when $r = k$.

Hence the probability of the occurrence of the associated pairs of x's and y's is the greatest when

$$k = r = \frac{\Sigma(xy)}{N\sigma_x\sigma_y}$$

We assumed that the values of x and y were associated and sought a value for r which would give the maximum probability that for any x we would obtain its associated y. This leads to a formula using $\Sigma(xy)$ which formula we proceed to develop.

The Product Moment Formula

(Forsyth - "Mathematical Analysis of Statistics")

If we define the product-moment of a surface $z = f(x, y)$ by the relation

$$\xi(x, y) = \iint xy\, f(x, y)\, dx\, dy$$

and if we take the correlation surface

$$z = k\, e^{-\frac{1}{2}(ax^2 + 2hxy + by^2)}$$

then the product moment of the correlation surface about z, the centroid vertical is:

(1) $\xi xy = k \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\, e^{-\frac{1}{2}(ax^2 + 2hxy + by^2)}\, dx\, dy$

(2)

(3) Now consider only $\int_{-\infty}^{\infty} x\, e^{-\frac{1}{2}(ax^2 + 2hxy + by^2)}\, dx$

(4) $\int_{-\infty}^{\infty} x\, e^{-\frac{1}{2}(ax^2 + 2hxy + by^2)}\, dx = -\frac{1}{a} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(ax^2 + 2hxy + by^2)} (-ax - hy + hy)\, dx$

(5) $= -\frac{1}{a} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(ax^2 + 2hxy + by^2)} (-ax - hy)\, dx - \frac{hy}{a} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(ax^2 + 2hxy + by^2)}\, dx$

(6) Now consider $-\frac{1}{a} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(ax^2 + 2hxy + by^2)} (-ax - hy)\, dx$

(7)

(8) $= -\frac{1}{a}\, e^{-\frac{a}{2}\left(\frac{by^2}{a} - \frac{h^2y^2}{a^2}\right)} \int_{-\infty}^{\infty} e^{-\frac{a}{2}\left(x^2 + \frac{2hxy}{a} + \frac{h^2y^2}{a^2}\right)} (-ax - hy)\, dx$

(9) Now consider only $\int_{-\infty}^{\infty} e^{-\frac{a}{2}\left(x^2 + \frac{2hxy}{a} + \frac{h^2y^2}{a^2}\right)} (-ax - hy)\, dx =$

$$\int_{-\infty}^{\infty} e^{-\frac{a}{2}\left(x + \frac{hy}{a}\right)^2} (-ax - hy)\, dx = 0$$

(10) Therefore, from (5) and (9)

$$\int_{-\infty}^{\infty} x e^{-\frac{1}{2}(ax^2+2hxy+by^2)}\,dx = -\frac{hy}{a}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(ax^2+2hxy+by^2)}\,dx$$

(11) $$-\frac{hy}{a}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(ax^2+2hxy+by^2)}\,dx = -\frac{hy}{a}\sqrt{\frac{2\pi}{a}}\,e^{-\frac{y^2}{2\sigma_y^2}}$$

shown in section 6 and 7 on page 25.

(12) So $$\int_{-\infty}^{\infty} x e^{-\frac{1}{2}(ax^2+2hxy+by^2)}\,dx = -\frac{hy}{a}\sqrt{\frac{2\pi}{a}}\,e^{-\frac{y^2}{2\sigma_y^2}}$$

(13) Substituting (12) in (2)

$$\Sigma(xy) = k\int_{-\infty}^{\infty} y \cdot -\frac{hy}{a}\sqrt{\frac{2\pi}{a}}\,e^{-\frac{y^2}{2\sigma_y^2}}\,dy$$

(14) Now

$$\Sigma(xy) = -\frac{hk\sqrt{2\pi}}{a\sqrt{a}}\int_{-\infty}^{\infty} y^2 e^{-\frac{y^2}{2\sigma_y^2}}\,dy$$

$$\int_{-\infty}^{\infty} y^2 e^{-\frac{y^2}{2\sigma_y^2}}\,dy = \int_{-\infty}^{\infty} \sigma_y^2 e^{-\frac{y^2}{2\sigma_y^2}}\,dy$$ integrating by parts.

(15) $$\therefore \Sigma(xy) = -\frac{hk\sqrt{2\pi}}{a\sqrt{a}}\int_{-\infty}^{\infty} \sigma_y^2 e^{-\frac{y^2}{2\sigma_y^2}}\,dy$$

(16) $$\int_{-\infty}^{\infty} \sigma_y^2 e^{-\frac{y^2}{2\sigma_y^2}}\,dy = \sigma_y^3\sqrt{2\pi}$$

(17) $$\therefore \Sigma(xy) = -\frac{2hk\pi}{a\sqrt{a}}\,\sigma_y^3$$

(18) In Chapter IV, we found on pages 25 and 26

$$a = \frac{1}{\sigma_x^2(1-r^2)}$$

$$h = \frac{-r}{\sigma_x\sigma_y(1-r^2)}$$

$$k = \frac{N}{2\pi\sigma_x\sigma_y\sqrt{1-r^2}}$$

(19) So

$$\Sigma(xy) = \frac{-2\cdot\frac{-r}{\sigma_x\sigma_y(1-r^2)}\cdot\frac{N}{2\pi\sigma_x\sigma_y\sqrt{1-r^2}}\cdot\pi\sigma_y^3}{\frac{1}{\sigma_x^2(1-r^2)}\cdot\frac{1}{\sigma_x\sqrt{1-r^2}}}$$

(20) $$\Sigma(xy) = Nr\sigma_x\sigma_y$$

(21) $$r = \frac{\Sigma(xy)}{N\sigma_x\sigma_y}$$

Chapter VI Some Interesting Points Arrived at by Considering Normal Correlation.

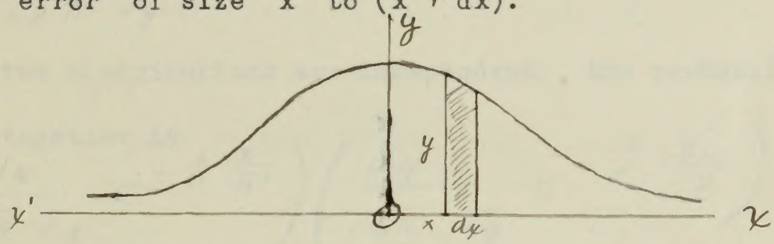
(Jones - "A First Course in Statistics"
Chapter XIX)

Frequency Surface for Two Correlated Variables

Before discussing the frequency surface for two correlated variables, it might be helpful to summarize briefly some important features of the normal curve of error.

The equation of the normal curve is $y = \frac{N}{\sqrt{2\pi}\,\sigma_x} e^{-\frac{x^2}{2\sigma_x^2}}$

where y dx measures the frequency with which a variable deviates from the mean by an amount lying between x and (x+dx) ; i.e. ydx measures the frequency of "error" of size x to (x + dx).



The probability of an error lying between x_1 and x_2 is given by the ratio.

$$\frac{\text{frequency of all errors between the given limits}}{\text{frequency of all errors}}$$

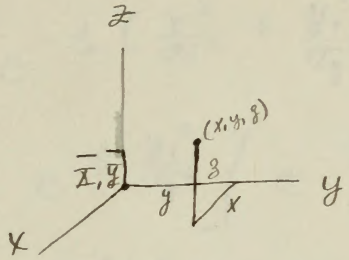
The frequency of all errors is $\int_{-\infty}^{\infty} y\,dx = N$

Therefore, the probability of frequency of errors between x and x +dx is $\frac{y\,dx}{N}$

$$= \frac{dx}{\sqrt{2\pi}\,\sigma_x} e^{-\frac{x^2}{\sigma_x^2}}$$

Frequency Surface, showing the distribution of two completely independent variables each subject to the normal law.

If $\bar{X}$, $\bar{y}$ be taken as the origin and x and y the deviations from $\bar{X}$ & $\bar{y}$ respectively,



then the probability of a deviation between x and x + dx is

$$\frac{dx}{\sqrt{2\pi}\,\sigma_x} e^{-\frac{1}{2}\frac{x^2}{\sigma_x^2}}$$

and the probability of a deviation between y and y + dy is

$$\frac{dy}{\sqrt{2\pi}\,\sigma_y} e^{-\frac{1}{2}\frac{y^2}{\sigma_y^2}}$$

Since the two distributions are independent , the probability of their occurring together is

$$\left(\frac{dx}{\sqrt{2\pi}\,\sigma_x} e^{-\frac{1}{2}\frac{x^2}{\sigma_x^2}}\right)\left(\frac{dy}{\sqrt{2\pi}\,\sigma_y} e^{-\frac{1}{2}\frac{y^2}{\sigma_y^2}}\right)$$

$$= \frac{dx\,dy}{2\pi\sigma_x\sigma_y} e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}\right)}$$

If N is the total number of observations, the frequency with which such deviations occur together is

$$\frac{N}{2\pi\sigma_x\sigma_y} e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}\right)} dx\,dy$$

If $z\,dx\,dy = \frac{N}{2\pi\sigma_x\sigma_y} e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}\right)} dx\,dy$

then $z = \frac{N}{2\pi\sigma_x\sigma_y} e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}\right)}$

This, then, is the equation of the frequency surface when the variables are independent.

Now let us discuss this surface to see what it is like.

If $y = y_1$

$$z = \frac{N}{2\pi\sigma_x\sigma_y} e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y_1^2}{\sigma_y^2}\right)}$$

$$= \left[\frac{N}{2\pi\sigma_x\sigma_y} e^{-\frac{1}{2}\frac{y_1^2}{\sigma_y^2}}\right] e^{-\frac{1}{2}\frac{x^2}{\sigma_x^2}}$$

$$= \frac{N_1}{\sqrt{2\pi}\,\sigma_x} e^{-\frac{1}{2}\frac{x^2}{\sigma_x^2}} \quad \text{where } N_1 = \frac{N}{\sqrt{2\pi}\,\sigma_y} e^{-\frac{1}{2}\frac{y_1^2}{\sigma_y^2}}$$

But $z = \frac{N_1}{\sqrt{2\pi}\,\sigma_x} e^{-\frac{1}{2}\frac{x^2}{\sigma_x^2}}$ is the equation of a normal curve in which $\bar{x}$, σ_x are not affected by different values of y. Hence all arrays of $\underline{X}$ are similar, having the same mean and the same standard deviation.

Since the surface is symmetrical, the same may be said for all the arrays of $\underline{y}$.

Furthermore, if $z = k$, a constant

$$k = \frac{N}{2\pi\sigma_x\sigma_y} e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right)}$$

$$\frac{2\pi k\sigma_x\sigma_y}{N} = e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right)}$$

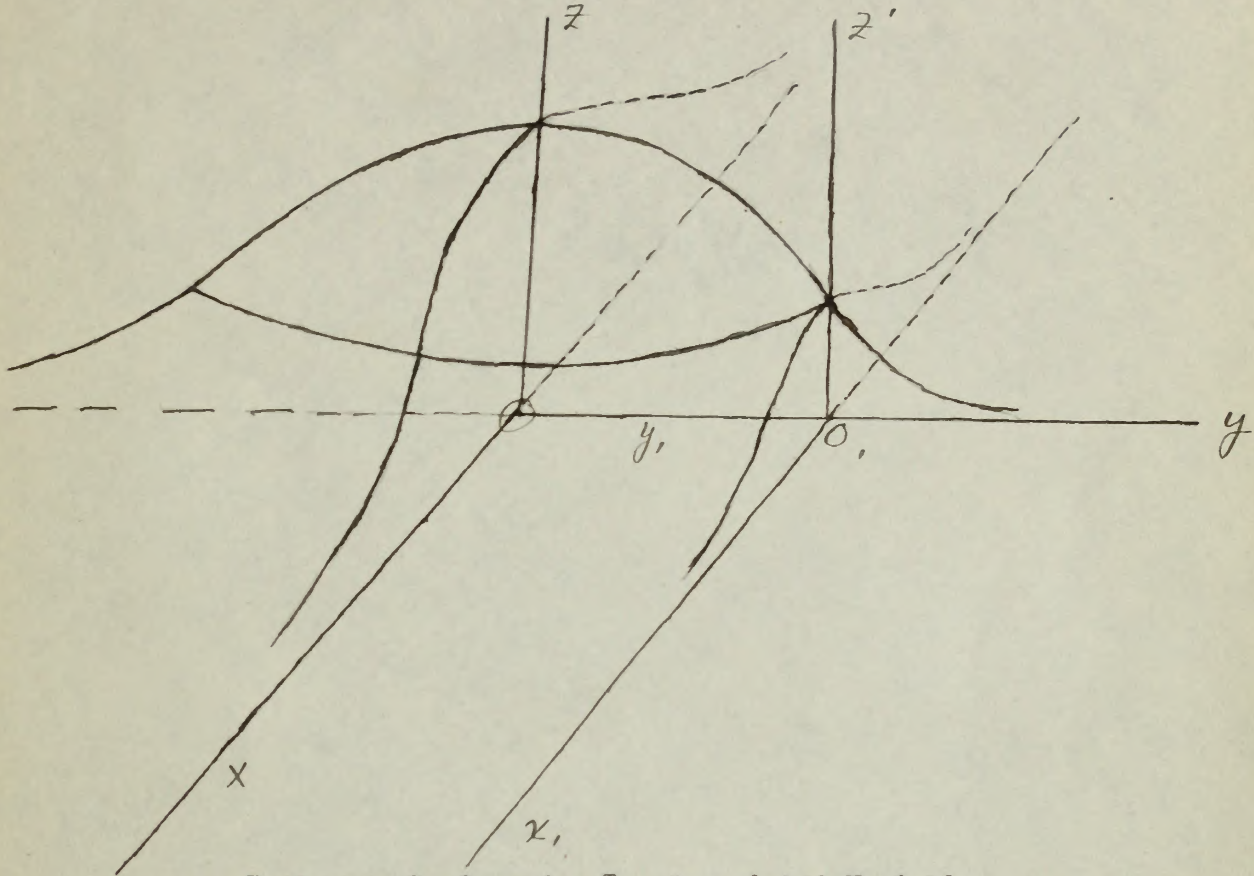
Since the left-hand side is a constant, the right-hand side is also, so

$$\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} = c, \quad \text{a constant.}$$

That is the equation of an ellipse; thus we may say where x and y occur with the frequency k, the points (x,y) lie on an ellipse in the plane $z = k$ or

$$\begin{cases} z = k \\ \frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} = c \end{cases}$$ defines the locus of

points where x and y occur with the same frequency. If we vary k, the frequency, and consequently vary c, we get a series of ellipses. If we project these orthogonally on the plane $\bar{X}\,O\,\bar{Y}$ we would get a series of concentric similar ellipses. This enables us to draw the surface



Frequency Surface for Two Correlated Variables.

If we consider the variables $\bar{X}$ and $\bar{Y}$ and take $\bar{\bar{X}}$, $\bar{\bar{Y}}$ as the origin so that $\bar{X} - \bar{\bar{X}} = x$ and $\bar{Y} - \bar{\bar{Y}} = y$, then the line of regression giving the best y corresponding to any x is $y = r \dfrac{\sigma_y}{\sigma_x} x$

If we consider η the error made in taking y from this equation instead of the observed y, then $\eta = y - r \dfrac{\sigma_y}{\sigma_x} x$. Thus for every (x, y) there is an η and the same η occurs as often as any pair (x, y) is repeated; thus the frequency distribution (x, η) is exactly the same as that of (x, y). The correlation between η and x is $\dfrac{\Sigma(x\eta)}{N\sigma_x\sigma_\eta}$ and should equal zero.

$$\Sigma(x\eta) = \Sigma\left[x\left(y - r\frac{\sigma_y}{\sigma_x}x\right)\right]$$

$$= \Sigma(xy) - r\frac{\sigma_y}{\sigma_x}\Sigma(x^2)$$

$$= NP - \frac{P}{\sigma_x\sigma_y}\cdot\frac{\sigma_y}{\sigma_x}\cdot N\sigma_x^2$$

$$= NP - NP$$

$$\Sigma(x\eta) = 0$$

Therefore, the variables x and η are independent and the probability of their occurring together is the product of their separate probabilities.

The probability of a deviation between x and $(x + dx)$ occurring if we consider x alone is

$$\frac{dx}{\sqrt{2\pi}\,\sigma_x}e^{-\frac{x^2}{2\sigma_x^2}}$$

and the probability of a deviation between η and $(\eta + d\eta)$ is

$$\frac{d\eta}{\sqrt{2\pi}\,\sigma_\eta}e^{-\frac{\eta^2}{2\sigma_\eta^2}}$$

The probability of a combined occurrence of these deviations is

$$\frac{dx\,d\eta}{2\pi\sigma_x\sigma_\eta}e^{-\frac{1}{2}\left\{\frac{x^2}{\sigma_x^2} + \frac{\eta^2}{\sigma_\eta^2}\right\}}$$

(1) $$= \frac{dx\,d\eta}{2\pi\sigma_x\sigma_\eta}e^{-\frac{1}{2}\left\{\frac{x^2}{\sigma_x^2} + \frac{\left(y - r\frac{\sigma_y}{\sigma_x}x\right)^2}{\sigma_\eta^2}\right\}}$$

(2) $$= \frac{dx\,d\eta}{2\pi\sigma_x\sigma_\eta}e^{-\frac{1}{2}\left[\frac{y^2}{\sigma_\eta^2} - \frac{2xyr\sigma_x}{\sigma_x\sigma_\eta^2} + x^2\left(\frac{1}{\sigma_x^2} + \frac{r^2\sigma_y^2}{\sigma_x^2\sigma_\eta^2}\right)\right]}$$

But $$N\sigma_\eta^2 = \Sigma\left(y - r\frac{\sigma_y}{\sigma_x}x\right)^2$$

$$= \Sigma y^2 - 2r\frac{\sigma_y}{\sigma_x}\Sigma(xy) + \frac{r^2\sigma_y^2}{\sigma_x^2}\Sigma(x^2)$$

$$= N\sigma_y^2 + Nr^2\sigma_y^2$$

(3) $$= N\sigma_y^2(1 - r^2)$$

Similarly if g is the error made in estimating any x from $X = \frac{r\sigma_x}{\sigma_y}y$ then

(4) $$N\sigma_g^2 = N\sigma_x^2(1 - r^2)$$

(5) $\dfrac{\sigma_g^{\,2}}{\sigma_x^{\,2}} = 1 - r^2 = \dfrac{\sigma_n^{\,2}}{\sigma_y^{\,2}}$

(6) $\text{So } \dfrac{\sigma_y}{\sigma_x \sigma_n^2} = \dfrac{1}{\sigma_x \sigma_n} \cdot \dfrac{\sigma_y}{\sigma_n} = \dfrac{1}{\sigma_x \sigma_n} \cdot \dfrac{\sigma_x}{\sigma_g} = \dfrac{1}{\sigma_n \sigma_g}$

(7) Also $\dfrac{1}{\sigma_x^2} + \dfrac{r^2 \sigma_y^2}{\sigma_x^2 \sigma_n^2} = \dfrac{1}{\sigma_x^2}\left(1 + r^2 \dfrac{\sigma_y^2}{\sigma_n^2}\right)$

$$= \frac{1}{\sigma_x^2}\left(1 + r^2 \cdot \frac{1}{1-r^2}\right)$$

$$= \frac{1}{\sigma_x^2} \cdot \frac{1}{1-r^2}$$

$$= \frac{1}{\sigma_g^2}$$

(8) Substituting (3), (4), (6) and (7) in (2)

$$= \frac{dx\, dn}{2\pi \sigma_x \sigma_y \sqrt{1-r^2}}\, e^{-\frac{1}{2}\left(\frac{y^2}{\sigma_n^2} - 2xyr\frac{1}{\sigma_g \sigma_n} + \frac{x^2}{\sigma_g^2}\right)}$$

This is the probability of the combined occurrence of deviations x to $(x+dx)$, n to $(n+dn)$. Now if we substitute (3) and (4) in (8) we get the frequency of the combined occurrence of deviations x to $(x+dx)$ and y to $(y+dy)$

(9) $$= \frac{dx\, dy}{2\pi \sigma_x \sigma_y \sqrt{1-r^2}}\, e^{-\frac{1}{2}\left(\frac{y^2}{\sigma_y^2(1-r^2)} - 2xyr\frac{1}{\sigma_x \sigma_y (1-r^2)} + \frac{x^2}{\sigma_x^2(1-r^2)}\right)}$$

$$= \frac{dx\, dy}{2\pi \sigma_x \sigma_y \sqrt{1-r^2}}\, e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - 2r\frac{xy}{\sigma_x \sigma_y}\right)\frac{1}{1-r^2}}$$

Thus if $z\,dx\,dy$ represents this frequency where N is the total number of observations

$$z = \frac{N}{2\pi \sqrt{1-r^2}\, \sigma_x \sigma_y}\, e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - 2r\frac{xy}{\sigma_x \sigma_y}\right)\frac{1}{1-r^2}}$$

This equation represents the frequency surface for two correlated variables.

Let us now study this equation in order to learn some of its interesting points.

Let t (a constant) = $\frac{N}{2\pi\sqrt{1-r^2}\,\sigma_x\sigma_y}$

and consider the surface

$$z = t\,e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - 2r\frac{xy}{\sigma_x\sigma_y}\right)\frac{1}{1-r^2}}$$

If we let $y = y_1$, we have

$$z = t\,e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y_1^2}{\sigma_y^2} - 2r\frac{xy_1}{\sigma_x\sigma_y}\right)\frac{1}{1-r^2}}$$

$$= t\,e^{-\frac{1}{2(1-r^2)}\left\{\frac{y_1^2}{\sigma_y^2}(1-r^2) + \left(\frac{x}{\sigma_x} - r\frac{y_1}{\sigma_y}\right)^2\right\}}$$

$$= t\cdot e^{-\frac{y_1^2}{2\sigma_y^2}}\cdot e^{-\frac{1}{2(1-r^2)}\left(\frac{x}{\sigma_x} - r\frac{y_1}{\sigma_y}\right)^2}$$

(1) $$z = \frac{t}{e^{\frac{y_1^2}{2\sigma_y^2}}}\cdot e^{-\frac{1}{2\sigma_x^2(1-r^2)}\left(x - ry_1\frac{\sigma_x}{\sigma_y}\right)^2}$$

This is the equation arrived at by taking the equation of the normal curve $z = \frac{t}{e^{\frac{y_1^2}{2\sigma_y^2}}}\,e^{-\frac{1}{2\sigma_x(1-r^2)}x^2}$, an equation in x and z in the plane $y = y_1$, and shifting through a distance $r\,y_1\frac{\sigma_x}{\sigma_y}$ along an axis parallel to σX. The equation is that of a normal curve with a standard deviation $\sigma_x\sqrt{1-r^2}$ and the mean at the intersection of the planes $y = y_1$ and $x = r\,y_1\frac{\sigma_x}{\sigma_y}$. So the greatest frequency in this particular distribution is $z = \frac{t}{e^{\frac{y_1^2}{2\sigma_y^2}}}$, determined by the intersection of the two planes above.

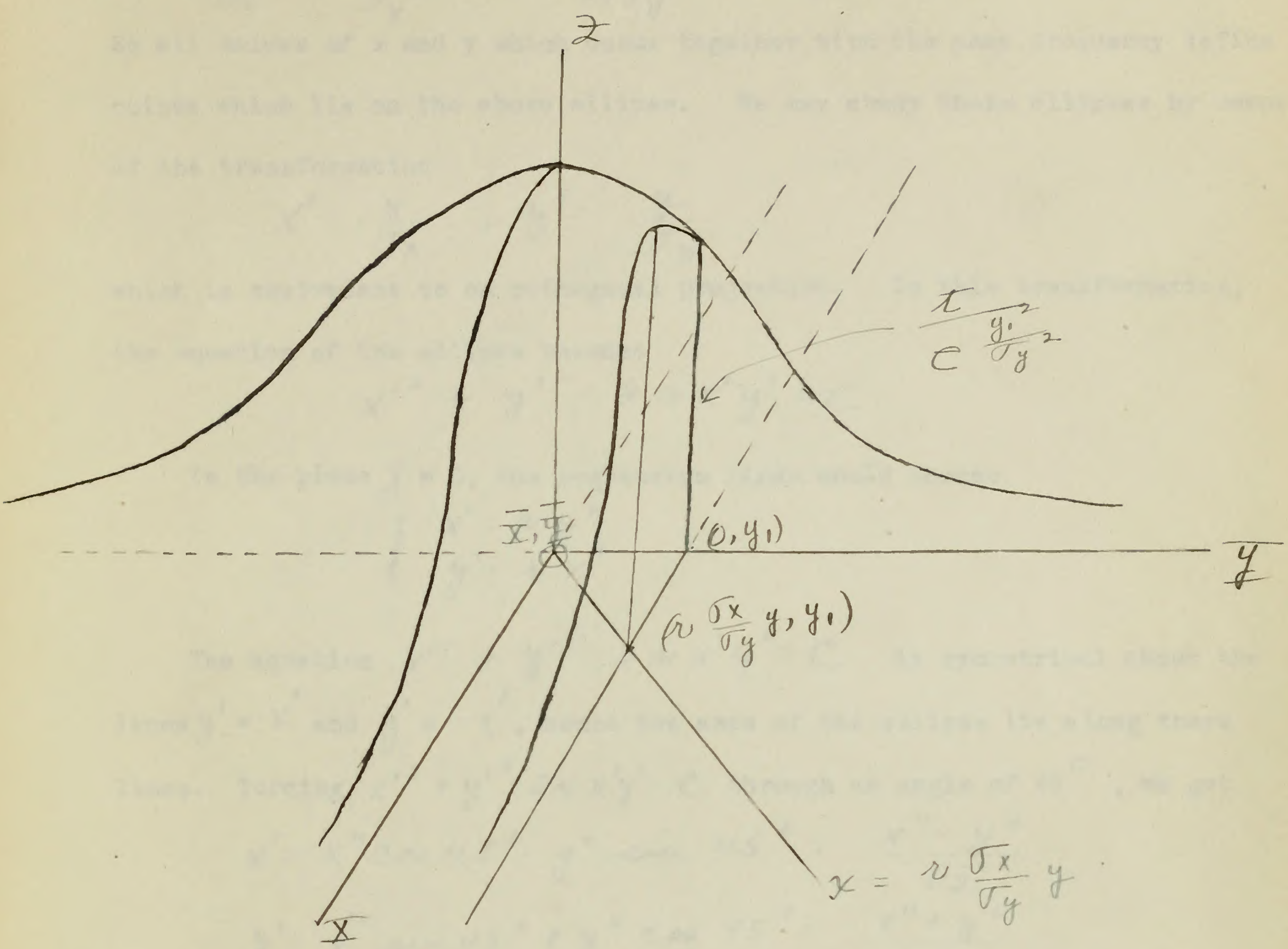
If $y = 0$

$$z = t\,e^{-\frac{1}{2(1-r^2)}\frac{x^2}{\sigma_x^2}}$$

This is a normal curve with a standard deviation $\sigma_x(1-r^2)$ and the mean at the origin where $z = t$. This mean may be considered the intersection of the plane $y = 0$ and $\frac{x}{\sigma_x} = r\frac{y}{\sigma_y}$.

Thus the planes giving the means of the x's corresponding to particular

values of y meet $z = 0$ in the regression line $\frac{x}{\sigma_x} = r\,\frac{y}{\sigma_y}$.

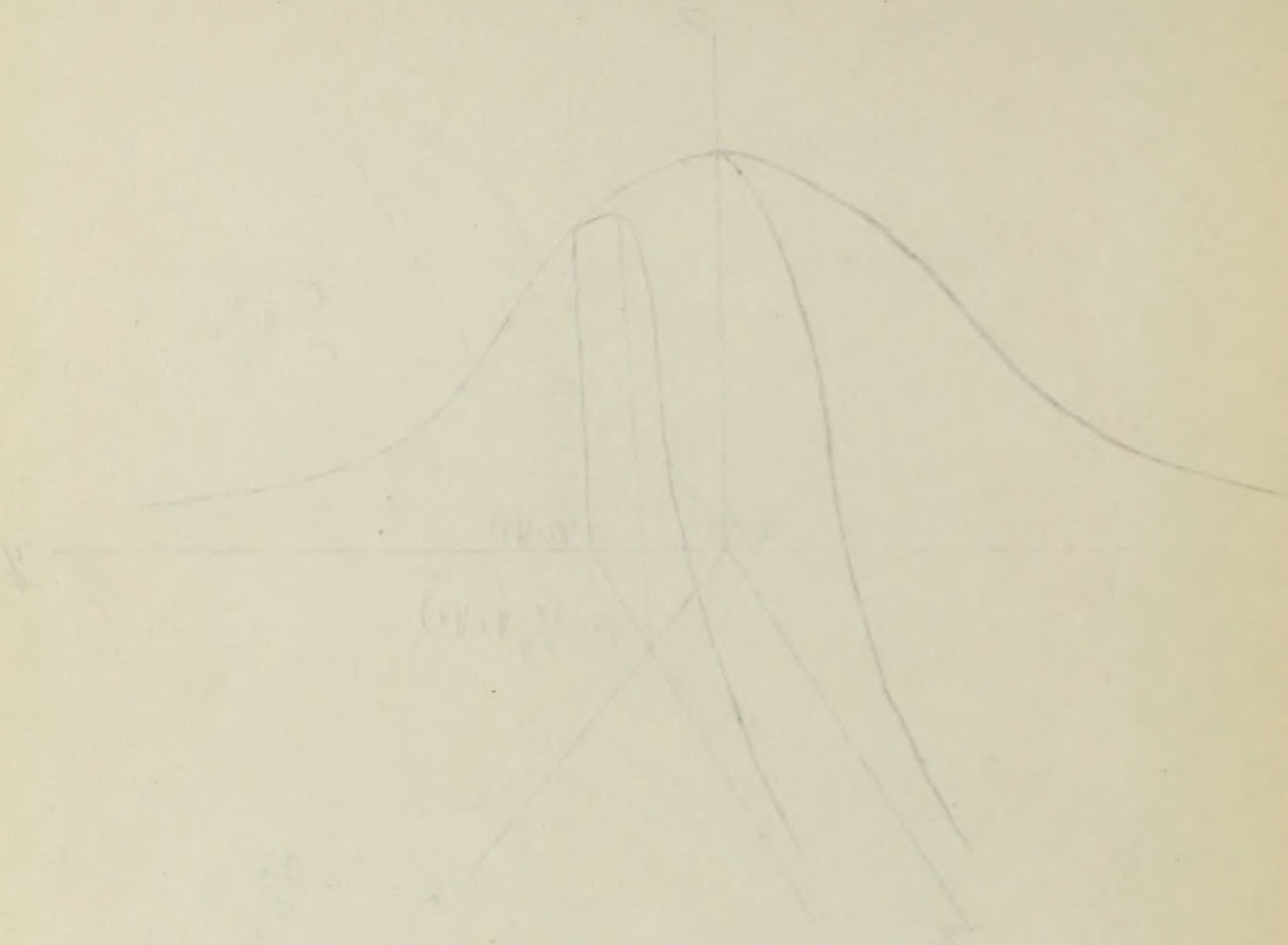


Thus the x arrays all have the same standard deviation $\sigma_x\sqrt{1-r^2}$, and all have their means at the intersections of the planes through particular values of y and the plane through the lines $z = 0$ and the regression line $x = r\frac{\sigma_x}{\sigma_y}y$.

Similarly it can be shown that all the y arrays are normal distributions, have the same standard deviation $\sigma_y\sqrt{1-r^2}$ and have their means at the intersection of the planes through particular values of x and the plane determined by the lines $z = 0$ and the regression line $y = r\frac{\sigma_y}{\sigma_x}x$

If we consider the equation

$$z = t\,e^{-\frac{1}{2(1-r^2)}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - 2r\frac{xy}{\sigma_x\sigma_y}\right)}$$



and let z equal a constant, k, then

$$\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - 2r\frac{xy}{\sigma_x \sigma_y} = C \quad \text{, a constant.}$$

So all values of x and y which occur together with the same frequency define points which lie on the above ellipse. We may study these ellipses by means of the transformation

$$x' = \frac{x}{\sigma_x} \quad , \quad y' = \frac{y}{\sigma_y}$$

which is equivalent to an orthogonal projection. In this transformation, the equation of the ellipse becomes

$$x'^2 + y'^2 - 2rx'y' = C$$

In the plane z = 0, the regression lines would become

$$\begin{cases} x' = ry' \\ y' = rx' \end{cases}$$

The equation $x'^2 + y'^2 - 2rx'y' = C$ is symmetrical about the lines $y' = x'$ and $y' = -x'$, hence the axes of the ellipse lie along these lines. Turning $x'^2 + y'^2 - 2rx'y' = C$ through an angle of 45° , we get

$$x' = x'' \cos 45^\circ - y'' \sin 45^\circ = \frac{x'' - y''}{\sqrt{2}}$$

$$y' = x'' \sin 45^\circ + y'' \cos 45^\circ = \frac{x'' + y''}{\sqrt{2}}$$

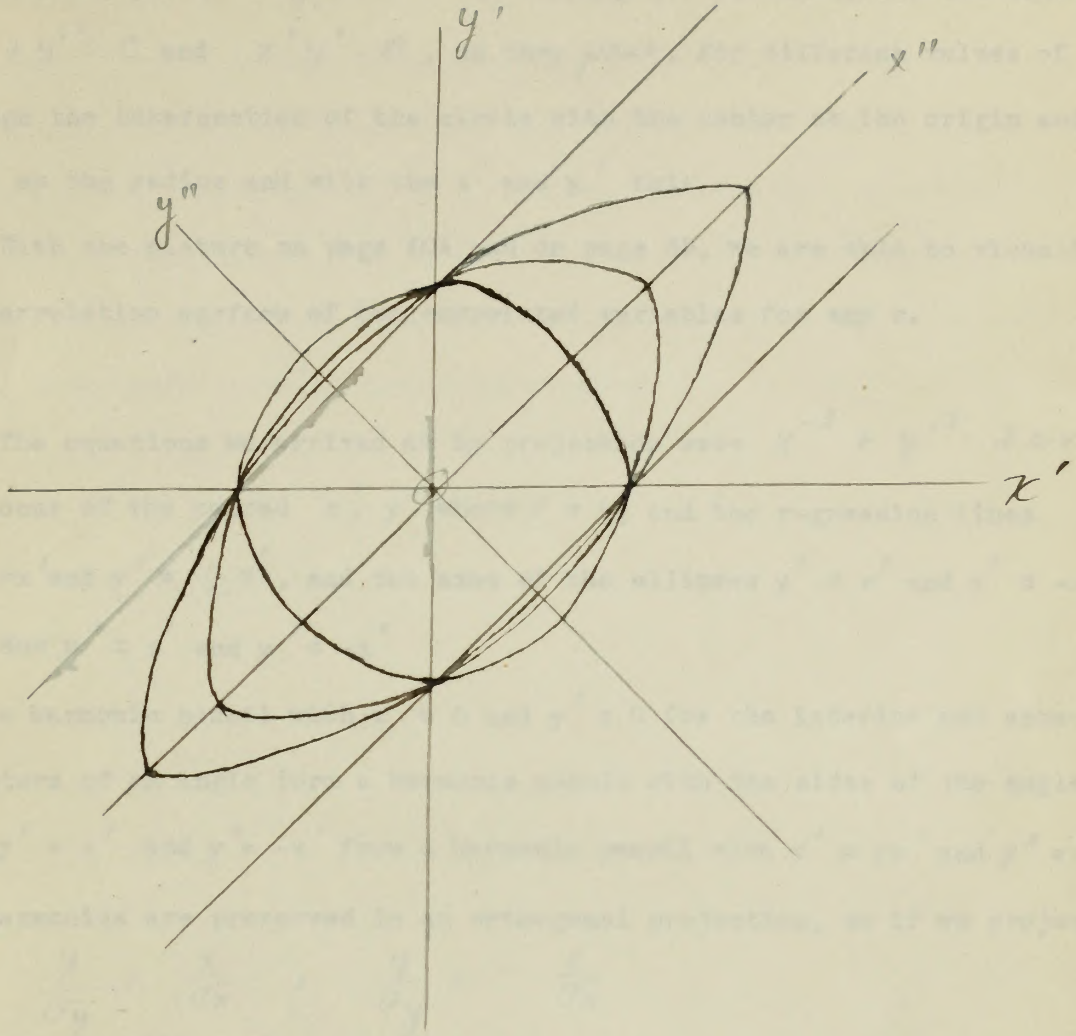
so the equation becomes

$$\frac{(x''-y'')^2}{2} + \frac{(x''+y'')^2}{2} - 2r\frac{(x''-y'')(x''+y'')}{2} = C$$

$$x''^2 + y''^2 - r(x''^2 - y''^2) = C$$

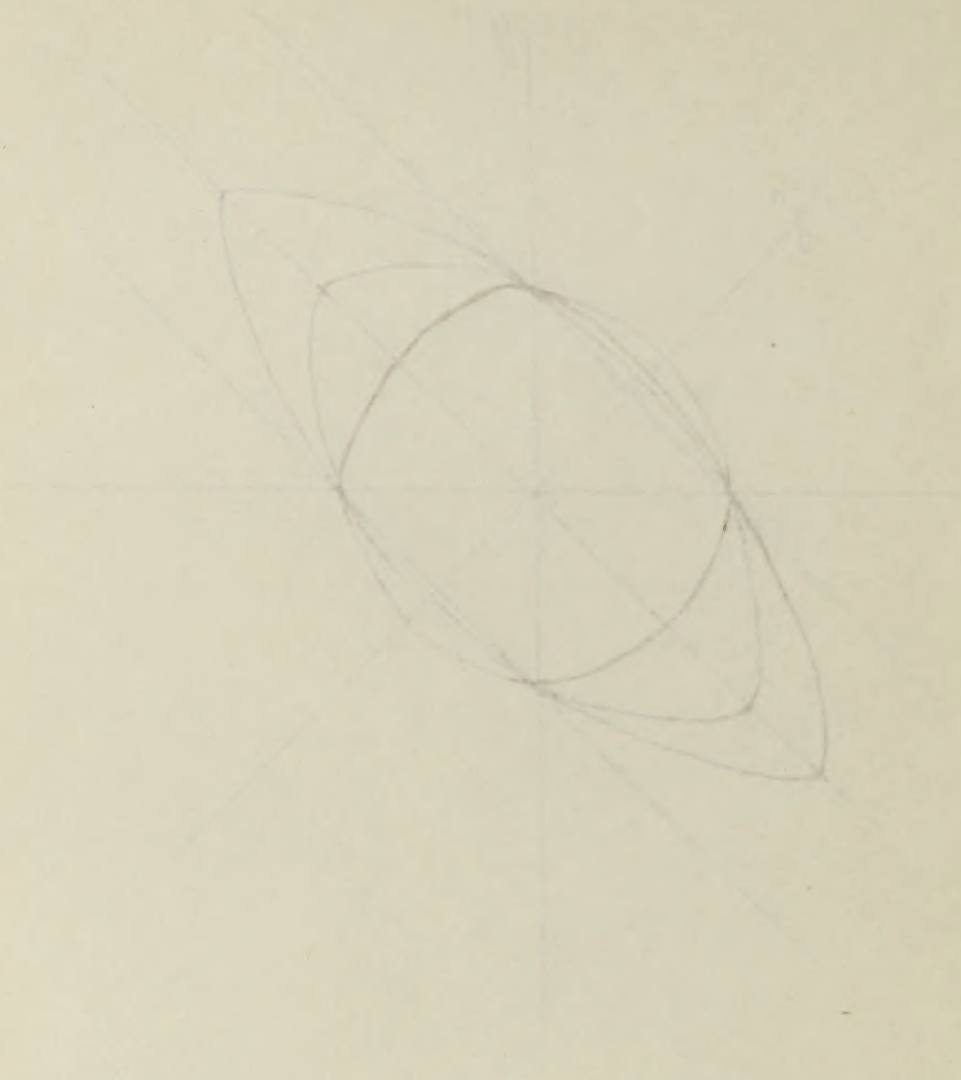
$$\frac{x''^2}{\frac{C}{1-r}} + \frac{y''^2}{\frac{C}{1+r}} = 1$$

Hence the semi-major axis is $\sqrt{\frac{C}{1-r}}$ and the semi-minor axis is $\sqrt{\frac{C}{1+r}}$. As r increases from 0 to 1, the semi-major axis increases from $\sqrt{C}$ to ∞ and the semi-minor axis decreases from $\sqrt{C}$ to $\sqrt{\frac{C}{2}}$; as r decreases from 0 to -1, the semi-major axis decreases from $\sqrt{C}$ to $\sqrt{\frac{C}{2}}$



Showing projections in the x y plane of the plane z = k

with correlation surfaces having different values of r.



and the semi-minor axis increases from $\sqrt{C}$ to ∞ .

The ellipse $x'^2 + y'^2 - 2rx'y' = C$ may be written as the two equations $x'^2 + y'^2 = C$ and $x'y' = 0$, so they *pass*, for different values of C, through the intersection of the circle with the center at the origin and $\sqrt{C}$ as the radius and with the x' and y' axis.

With the picture on page 40A and on page 39, we are able to visualize the correlation surface of two correlated variables for any r.

The equations we arrived at by projection were $x'^2 + y'^2 - 2rx'y' = C$, the locus of the paired x', y' where $r = k$, and the regression lines $y' = rx'$ and $y' = \frac{1}{r}x'$, and the axes of the ellipses $y' = x'$ and $y' = -x'$.

Now $y' = x'$ and $y' = -x'$

form a harmonic pencil with $x' = 0$ and $y' = 0$ for the interior and exterior bisectors of an angle form a harmonic pencil with the sides of the angle. Also $y' = x'$ and $y' = -x'$ form a harmonic pencil with $y' = rx'$ and $y' = \frac{1}{r}x'$. Now harmonics are preserved in an orthogonal projection, so if we project back,

$$\frac{y}{\sigma_y} = \frac{x}{\sigma_x} , \quad \frac{y}{\sigma_y} = -\frac{x}{\sigma_x}$$

are harmonic with

$$x = 0 , \quad y = 0$$

and

$$\frac{y}{\sigma_y} = \frac{x}{\sigma_x} , \quad \frac{y}{\sigma_y} = -\frac{x}{\sigma_x}$$

are harmonic with

$$\frac{y}{\sigma_y} = r\frac{x}{\sigma_x} \quad \text{and} \quad \frac{y}{\sigma_y} = \frac{1}{r}\frac{x}{\sigma_x}$$

Thus we say the two lines of regression corresponding to maximum correlation (r = +1, r = -1) are harmonic with

(1) The axes,

(2) The lines of regression for any r.

In other words, the lines of regression corresponding to maximum correlation bisect the interior and exterior angles formed by the lines of regression for any r, a fact which we have proved in a previous section.

We have shown that in an ideal distribution the means of the rows and the means of the columns all lie on the regression lines and in our previous work we have generalized this by assuming that the distributions we have had were so chosen that if there were an infinite number of paired values of x and y, they would form an ideal distribution.

Chapter VII Computation Formulas for the Coefficient of Correlation. Problems.

Having developed the formula

$$r = \frac{\sum_{i=1}^{n} x_i y_i}{N \sigma_x \sigma_y}$$

where

$$x_i = X_i - \bar{X}$$

$$y_i = Y_i - \bar{Y}$$

we may rewrite this in several ways which will aid in the numerical computation.

(1) $$\boxed{r = \frac{\sum_{i=1}^{n} x_i y_i}{N \sigma_x \sigma_y}}$$

Formula I

Origin at the true mean.

(2) $$r = \frac{\frac{1}{N} \sum xy}{\sqrt{\frac{1}{N}\sum(x^2)}\sqrt{\frac{1}{N}\sum(y^2)}}$$ by substituting $\sigma_x = \sqrt{\frac{1}{N}\sum(x^2)}$, $\sigma_y = \sqrt{\frac{1}{N}\sum(y^2)}$

so $$\boxed{r = \frac{\sum xy}{\sqrt{\sum(x^2)}\sqrt{\sum(y^2)}}}$$

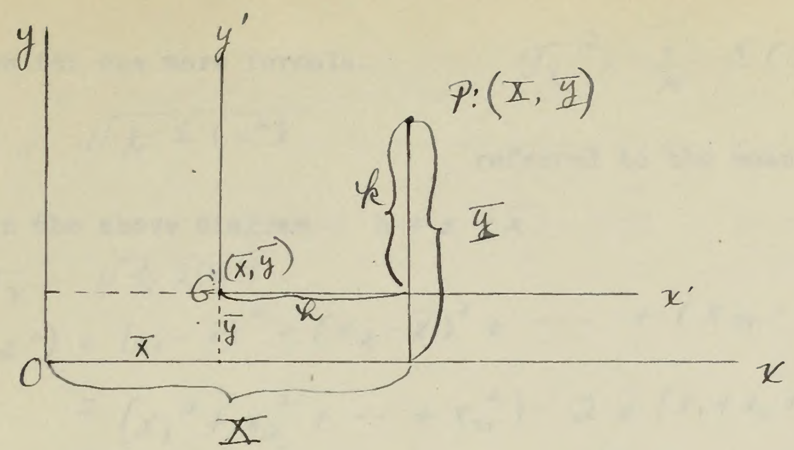
Formula II

Origin at the true mean.

(3) $$\boxed{r = \frac{\frac{1}{N}\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sigma_x \sigma_y}}$$ where $x_i = X_i - \bar{X}$, $y_i = Y_i - \bar{Y}$

Formula III Origin at True Mean.

(4) Now to rewrite this formula so that the deviations will refer to some point other than the true mean as origin.



Let h = $x - \bar{x}$

k = $y - \bar{y}$

In the formula $r = \frac{\sum x_i y_i}{N \sigma_x \sigma_y}$ x and y represented the deviations from the mean, and in the above h and k are the respective deviations, so we may rewrite

$$r = \frac{\sum h k}{N \sigma_x \sigma_y}$$

$$\sum(h k) = (x_1 - \bar{x})(y_1 - \bar{y}) + (x_2 - \bar{x})(y_2 - \bar{y}) + \cdots + (x_n - \bar{x})(y_n - \bar{y})$$

$$= (x_1 y_1 - \bar{x} y_1 - x_1 \bar{y} + \bar{x}\bar{y}) + \cdots + (x_n y_n - \bar{x} y_n - x_n \bar{y} + \bar{x}\bar{y})$$

$$= (x_1 y_1 + x_2 y_2 + \cdots + x_n y_n) - \bar{x}(y_1 + y_2 + \cdots + y_n) - \bar{y}(x_1 + x_2 + \cdots + x_n) + n\bar{x}\bar{y}$$

$$= \sum(x_i y_i) - \bar{x} \sum y_i - \bar{y} \sum x_i + n \bar{x}\bar{y}$$

But $\frac{\sum y_i}{n} = \bar{y}$ and $\frac{\sum x_i}{n} = \bar{x}$

$$\therefore \sum(h k) = \sum(x_i y_i) - \bar{x}\, n\bar{y} - \bar{y}\, n\bar{x} + n\bar{x}\bar{y}$$

$$\sum(h k) = \sum(x_i y_i) - n\bar{x}\bar{y}$$

So

$$r = \frac{\sum h k}{N \sigma_x \sigma_y} = \frac{\sum(xy) - n\bar{x}\bar{y}}{N \sigma_x \sigma_y}$$

(5) $$\boxed{r = \frac{\frac{1}{N}\sum(xy) - \bar{x}\bar{y}}{\sigma_x \sigma_y}}$$

Formula IV.

Origin at an arbitrary point.

Now for one more formula. $\sigma_x^2 = \frac{1}{N} \Sigma(x^2)$

$$\sigma_x = \sqrt{\frac{1}{N}\Sigma(x^2)}$$

referred to the mean as origin.

Or if in the above diagram h = x $-\bar{x}$

$$\sigma_x = \sqrt{\frac{1}{N}\Sigma(h^2)}$$

$$\Sigma(h^2) = (x_1-\bar{x})^2 + (x_2-\bar{x})^2 + \cdots + (x_n-\bar{x})^2$$

$$= (x_1^2 + x_2^2 + \cdots + x_n^2) - 2\bar{x}(x_1 + x_2 + \cdots + x_n) + n\bar{x}^2$$

$$= \Sigma(x_i^2) - 2\bar{x}\,\Sigma(x_i) + n\bar{x}^2$$

$$= \Sigma(x_i^2) - 2\bar{x}\,n\bar{x} + n\bar{x}^2$$

~~But~~

$$\Sigma(h^2) = \Sigma(x_i^2) - n\bar{x}^2$$

$$\therefore \sigma_x = \sqrt{\frac{1}{N}\Sigma(x^2) - \bar{x}^2}$$

Similarly

$$\sigma_y = \sqrt{\frac{1}{N}\Sigma(y^2) - \bar{y}^2}$$

$$\boxed{\therefore r = \frac{\frac{1}{N}\Sigma(xy) - \bar{x}\bar{y}}{\sqrt{\frac{1}{N}\Sigma(x^2) - \bar{x}^2}\sqrt{\frac{1}{N}\Sigma(y^2) - \bar{y}^2}}}$$

Formula V

Origin at an arbitrary point.

Summary of Formulas for r.

Where x and y are deviations from the true mean.

I $\quad r = \dfrac{\frac{1}{N}\Sigma xy}{\sigma_x \sigma_y}$

II $\quad r = \dfrac{\Sigma xy}{\sqrt{\Sigma(x^2)}\sqrt{\Sigma(y^2)}}$

Where x and y are deviations from an assumed mean.

III $\quad r = \dfrac{\frac{1}{N}\Sigma(x-\bar{x})(y-\bar{y})}{\sigma_x \sigma_y}$

IV $\quad r = \dfrac{\frac{1}{N}\Sigma(xy) - \bar{x}\bar{y}}{\sigma_x \sigma_y}$

V $\quad r = \dfrac{\frac{1}{N}\Sigma(xy) - \bar{x}\bar{y}}{\sqrt{\frac{1}{N}\Sigma(x^2) - \bar{x}^2}\sqrt{\frac{1}{N}\Sigma(y^2) - \bar{y}^2}}$

Problem: Correlation of the Scores Received by 106 Reading School Pupils in the Henmon-Nelson and The Terman I. Q. Tests.

Pupil	Henmon-Nelson	Terman	Pupil	Henmon-Nelson	Terman
1	149	134	54	107	106
2	144	122	55	107	114
3	141	132	56	107	109
4	139	142	57	106	112
5	136	125	58	106	116
6	135	139	59	106	104
7	135	139	60	105	117
8	134	121	61	105	114
9	132	123	62	104	117
10	131	122	63	104	130
11	130	122	64	104	119
12	130	124	65	103	109
13	130	123	66	103	110
14	129	131	67	103	109
15	128	126	68	103	112
16	127	105	69	102	109
17	126	132	70	102	125
18	125	120	71	102	112
19	125	114	72	102	101
20	124	124	73	101	103
21	123	130	74	101	112
22	121	119	75	101	112
23	120	111	76	101	109
24	120	118	77	100	98
25	120	117	78	99	100
26	120	120	79	99	110
27	117	128	80	98	99
28	117	102	81	98	104
29	117	108	82	98	.117
30	116	125	83	98	106
31	115	126	84	97	110
32	115	114	85	96	92
33	115	112	86	95	97
34	114	122	87	95	97
35	114	118	88	95	124
36	113	120	89	94	103
37	113	119	90	94	109
38	112	126	91	91	101
39	112	123	92	91	104
40	112	110	93	91	105
41	111	120	94	90	100
42	111	111	95	90	104
43	111	109	96	90	110
44	110	115	97	88	95
45	110	104	98	87	93
46	110	117	99	87	99
47	110	121	100	87	97
48	109	127	101	84	91
49	109	121	102	80	105
50	109	106	103	80	94
51	109	112	104	77	80
52	108	114	105	77	83
53	108	102	106	75	84

Correlation Between Henmon - Nelson I.Q. and Terman Group Test I.Q. Given to 106 Reading School Children.

Henmon-Nelson	75-79	80-84	85-89	90-94	95-99	100-104	105-109	110-114	115-119	120-124	125-129	130-134	135-139	140-144	145-149	f_y	y	$f_y y$	$f_y y^2$	Σxy
Terman	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8					
145-149																	7			
140-144													1			1	6	6	36	36
135-139													2		1	3	5	15	75	100
130-134						1				1	1			1		4	4	16	64	52
125-129						1	1	1	3		2		1			9	3	27	81	60
120-124					1		1	5		2	1	6		1		17	2	34	68	100
115-119					1	2	2	4		3						12	1	12	12	9
110-114				1	2	5	5	2	2	1	1					19	0	0	0	0
105-109		1		2	1	4	3	1	1		1					14	-1	-14	14	10
100-104				5	2	2	2	1	1							13	-2	-26	52	36
95-99			3		3	1										7	-3	-21	63	57
90-94		2	1		1											4	-4	-16	64	64
85-89																	-5	0	0	0
80-84	3															3	-6	-18	108	108
75-79																				
f_x	3	3	4	8	11	16	14	14	7	7	6	6	4	2	1	N= 106	$\Sigma(y) = 15$			$\Sigma xy = 632$
x	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8		$\Sigma(y^2) = 638$			
$f_x x$	-18	-15	-16	-24	-22	-16	0	14	14	21	24	30	24	14	8		$\Sigma(x) = +38$			
$f_x x^2$	108	75	64	72	44	16	0	14	28	63	96	150	144	98	64		$\Sigma(x^2) = 1036$			

$$\bar{x} = \frac{38}{106} = 0.36 \qquad \bar{x}^2 = .1296$$

$$\bar{y} = \frac{15}{106} = 0.14 \qquad \bar{y}^2 = .0196$$

$$r = \frac{\frac{1}{N}\Sigma xy - \bar{x}\,\bar{y}}{\sigma_x \sigma_y}$$

$$\sigma_x = \sqrt{\frac{1}{N}\Sigma(x^2) - \bar{x}^2} = \sqrt{\frac{1036}{106} - .1296} = 3.10$$

$$\sigma_y = \sqrt{\frac{1}{N}\Sigma(y^2) - \bar{y}^2} = \sqrt{\frac{638}{106} - .0196} = 2.45$$

$$r = \frac{\frac{632}{106} - 0.36 \times 0.14}{3.10 \times 2.45}$$

$$r = \frac{5.96 - 0.050}{7.60}$$

$$r = +0.77+$$

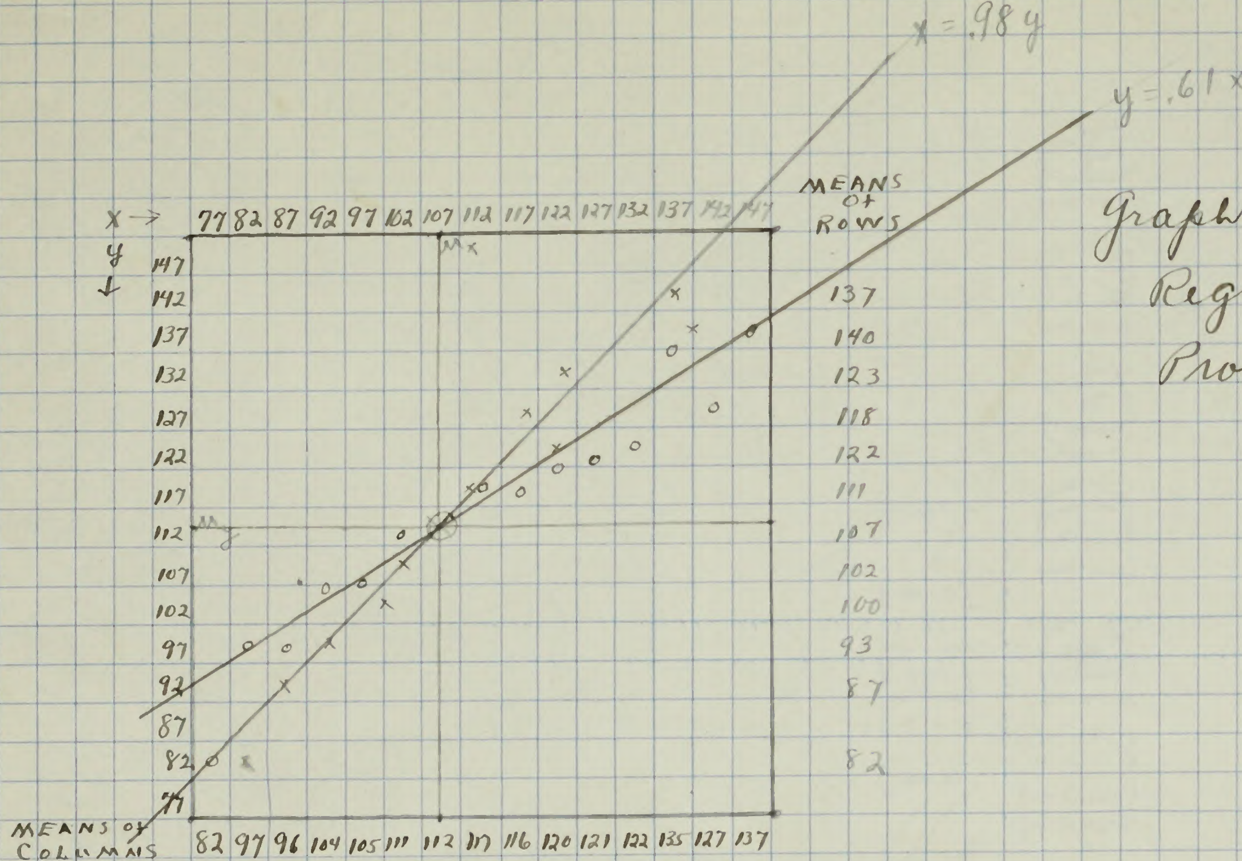
The regression lines.

$$x = r\frac{\sigma_x}{\sigma_y}\,y \qquad y = r\frac{\sigma_y}{\sigma_x}\,x$$

$$x = \frac{.77 \times 3.1}{2.45}\,y \qquad y = \frac{.77 \times 2.45}{3.1}$$

$$x = .98\,y \qquad y = .61\,x$$

Graph Showing Regression Lines For Problem on Page 46



Means of Columns o

y = .61 x line of best fit for means of Columns

Mean of x's = 107.36

Means of Rows x

Mean of y's = 112.13

x = .98 y line of best fit for means of Rows

Chapter VIII Correlation From Ranks

I. Introduction

When the data we are using expresses the measurements merely by the order or rank of the individual in the series, the product-moment formula for correlation is of no service in determining a measure of relationship. For example, consider the following table showing the ranks of ten students in an English and a history test.

	A	B	C	D	E	F	G	H	I	J
Rank in English test	1	2	3	4	5	6	7	8	9	10
Rank in History test	2	3	4	7	6	1	5	10	8	9
Differences in Rank	1	1	1	3	1	−5	−2	2	−1	−1

We will try to show that if D is the difference in ranks of corresponding variables in the two series of N individuals then the correlation between the ranks is given by $\rho = 1 - \dfrac{6 \Sigma D^2}{N(N^2-1)}$

Now it can be seen easily that correlation between actually measured variables can be made to change without changing ranks. For example, consider these series:

Variates

x	−2	−1	1	2
y	−2	−1	1	2

Ranks

1	2	3	4
1	2	3	4

The correlation of the variables and the correlation of the ranks are perfect.

Variates

x	−2	−1.9	1.9	2
y	−2	−.01	.01	2

Ranks

1	2	3	4
1	2	3	4

Here the correlation of the ranks is still perfect but not so the correlation of the variates.

This would indicate that the value of ρ is not worth much by itself for interpretation and would show the necessity of connecting ρ with r, the coefficient of correlation of the variates. We will, therefore, try to show that under the assumption of a normal frequency distribution, and the assumption that grades may be replaced by ranks, the corresponding value of the correlation coefficient of the variates that correspond to the ranks is given by

$$r = 2 \sin \left(\frac{\pi}{6} \rho \right)$$

II. The Formula $\rho = 1 - \dfrac{6 \Sigma D^2}{N(N^2-1)}$

Reference: T.L.Kelley "Statistical Methods" P. 191-4

If x and y be the deviations from the mean of two variables to be correlated and if

$$\sigma_D^2 = \frac{\Sigma (x-y)^2}{N}$$

then $\sigma_D^2 = \sigma_x^2 - 2r\sigma_x\sigma_y + \sigma_y^2$

$$r = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_D^2}{2\sigma_x\sigma_y}$$

If

$$\sigma_x = \sigma_y, \text{ then } r = 1 - \frac{\sigma_D^2}{2\sigma^2}$$

Now if we are considering the coefficient of correlation between two series measured in rank only, each series contains N terms, the standard deviations and the means of each are equal respectively. The difference between the actual ranks of any one character would be equal to the differences of their deviations from the mean, so we may use the above formula where the coefficient of correlation for ranks is defined as ρ.

$$\therefore \rho = 1 - \frac{\Sigma D^2}{2N\sigma^2}$$

Now to show

$$\sigma^2 = \frac{N^2 - 1}{12}$$

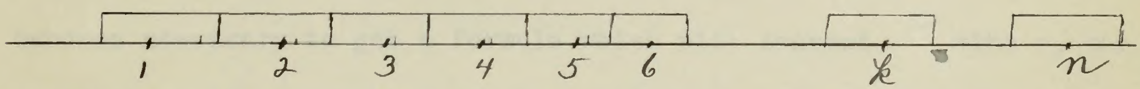
On Page 1 in the notes we show

$$S^2 = \sigma^2 + d^2$$

where S is the standard deviation about an arbitrary origin other than the mean. In this case let this origin be zero, then

$$d = \frac{1 + 2 + 3 + \cdots + N}{N} = \frac{N+1}{2}$$

S^2 is really the second moment of the ranks about zero, so we may determine S^2 by first determining $\overline{m}_2$, the second moment about zero where the distribution consists of a frequency evenly spread over the class intervals, as shown below, instead of being concentrated at the midpoints as is the case where rank positions are used.



The frequency distribution drawn is represented by the line y = 1 from $x = \frac{1}{2}$ to $x = N + \frac{1}{2}$. The second moment of any one rank, k, from 0 is k^2, whereas the second moment of the distribution y = 1 from $k - \frac{1}{2}$ to $k + \frac{1}{2}$ is

$$\int_{k-\frac{1}{2}}^{k+\frac{1}{2}} y\, x^2 dx = \frac{x^3}{3}\Big]_{k-\frac{1}{2}}^{k+\frac{1}{2}} = k^2 + \frac{1}{12}$$

The second moment of the frequency y = 1 corresponding to the k^{th} rank, $\frac{1}{N}$ of the frequency, is $\frac{1}{12}$ too large, as is true for every rank; hence the second moment of the equation y = 1 from $x = \frac{1}{2}$ to $x = N + \frac{1}{2}$ will be larger than the desired second moment by $\frac{1}{N}\left(\frac{N}{12}\right)$ or $\frac{1}{12}$. That is $\overline{m}_2 = S^2 + \frac{1}{12}$

$$\overline{m}_2 = \frac{1}{N}\int_{\frac{1}{2}}^{N+\frac{1}{2}} y x^2\, dy = \frac{4N^2+6N+3}{12}$$

$$S^2 = \frac{4N^2+6N+2}{12}$$

$$\therefore \sigma^2 = S^2 - d^2$$

$$= \frac{4N^2+6N+2}{12} - \left(\frac{N+1}{2}\right)^2$$

$$\sigma^2 = \frac{N^2-1}{12}$$

$$\therefore \rho = 1 - \frac{6\Sigma D^2}{N(N^2-1)}$$

III. The Formula $r = 2\sin\left(\frac{\pi}{6}\rho\right)$

Reference: Karl Pearson "On Further Methods of Determining Correlation" Drapers Company Research Memoirs Biometric Series IV.

Having found a formula for ρ for the correlation of ranks, it now becomes necessary to get a formula which will connect ρ with r, the coefficient of correlation for the variates. In the reference above Pearson develops such a formula

$$r = 2\sin\left(\frac{\pi}{6}\rho\right)$$

where, however, ρ is the correlation for grades and not ranks. The rank is the actual position in order of an individual and is assumed to be at the midpoint of the class interval, hence if the rank is k, there are k - $\frac{1}{2}$ individuals above that particular one in the series. Thus the grade of this particular individual would be k - $\frac{1}{2}$, the actual number above it in the series. Ranks form a discontinuous series with an interval of 1 while grades form a continuous series. The formula above may be used with ranks on the basis of two assumptions:

(1) The series we are dealing with follow the normal law.

(2) Grades of an individual may be replaced by ranks.

If we consider a series of N terms and each of these has a value in

an x and one in a y frequency and we wish to find the coefficient of correlation, we may let m_1, m_2 be the means; σ_1, σ_2 the standard deviations; r be the correlation; and x and y the deviations from the means respectively. Then Pearson defines

$$g_1 = \frac{1}{2}N + \frac{N}{2\pi\sigma_1}\int_0^x e^{-\frac{1}{2}\frac{x^2}{\sigma_1^2}}\,dx$$

$$g_2 = \frac{1}{2}N + \frac{N}{2\pi\sigma_2}\int_0^y e^{-\frac{1}{2}\frac{y^2}{\sigma_2^2}}\,dy$$

where g_1 and g_2 are the x and y grades of the individuals in the series and $\frac{1}{2}$ N is the mean of each. Since g_1 and g_2 are functions of x and y, correlation between g_1 and g_2 determines the correlation between x and y and vice versa.

Now if $i_1 = g_1 - \bar{g}_1$; $i_2 = g_2 - \bar{g}_2$;

and

$$z = \frac{1}{2\pi\sigma_1\sigma_2} \cdot \frac{1}{\sqrt{1-r^2}}\, e^{-\frac{1}{2(1-r^2)}\left(\frac{x^2}{\sigma_1^2} - \frac{2rxy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2}\right)}$$

then the product-moment of the grades is

(1) $$\rho_{g_1 g_2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} i_1 i_2 z\,dx\,dy$$

(2) $$\frac{d\rho_{g_1 g_2}}{dr} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} i_1 i_2 \frac{dz}{dr}\,dx\,dy$$

Pearson has shown in "Philosophical Transactions" Vol. 195A Page 25 that $\frac{dz}{dr} = \sigma_1\sigma_2 \frac{d^2z}{dx\,dy}$. The proof of this required the definitions and notations for multiple correlation, so it has been assumed in this paper.

(3) $$\frac{d\rho_{g_1 g_2}}{dr} = \sigma_1\sigma_2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} i_1 i_2 \frac{d^2z}{dx\,dy}\,dx\,dy$$

(4) Integrating twice by parts. (See Notes, Page 2)

$$\frac{d\rho_{g_1 g_2}}{dr} = \sigma_1\sigma_2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z\,\frac{di_1}{dx}\,\frac{di_2}{dy}\,dx\,dy$$

(5) Substituting for $\frac{di_1}{dx}$ and $\frac{di_2}{dy}$ their values and letting

$x = x'\sigma_1,\ y = y'\sigma_2$ (See Notes, Page 3)

$$\frac{d\,P_{g_1 g_2}}{dr} = \frac{N^3}{4\pi^2\sqrt{1-r^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2(1-r^2)}\{(2-r^2)x'^2 - 2rx'y' + (2-r^2)y'^2\}}\,dx'dy'$$

$$= \frac{N^3}{4\pi^2\sqrt{1-r^2}} \cdot \frac{1}{\sqrt{\left(\frac{2-r^2}{1-r^2}\right)^2 - \frac{r^2}{(1-r^2)^2}}}$$ (See Notes, Page 4)

$$= \frac{N^3}{2\pi\sqrt{4-r^2}}$$

(6) Defining $\rho = \frac{P_{g_1 g_2}}{N\sigma_{g_1}\sigma_{g_2}}$, $\left\{\begin{array}{l}\text{to correspond to the product-}\\ \text{moment formula for correlation}\end{array}\right.$

$$\frac{d\rho}{dr} = \frac{1}{N\sigma_{g_1}\sigma_{g_2}} \frac{d\,P_{g_1 g_2}}{dr}$$

$$\frac{d\rho}{dr} = \frac{6}{\pi} \cdot \frac{1}{\sqrt{4-r^2}}$$

(7) $\rho = \frac{6}{\pi} \sin^{-1}\frac{1}{2}r +$ a constant.

But since r is the coefficient of correlation between x and y and ρ is the coefficient of correlation between g_1 and g_2, ρ is zero when r is zero; therefore the constant above is zero.

(8) $\therefore\ r = 2\sin\left(\frac{\pi}{6}\rho\right)$

IV Problem Showing Correlation From Ranks Between Ten Students in a History and an English Examination.

Student	Rank in English Test	Rank in History Test	Difference	Square of Difference
A	1	2	1	1
B	2	3	1	1
C	3	4	1	1
D	4	7	3	9
E	5	6	1	1
F	6	1	-5	25
G	7	5	-2	4
H	8	10	2	4
I	9	8	-1	1
J	10	9	-1	1
				48

$$\rho = 1 - \frac{6 \Sigma D^2}{N(N^2-1)}$$

$$\Sigma D^2 = 48, \qquad 6 \Sigma D^2 = 288$$

$$N = 10, \qquad N(N^2-1) = 990$$

$$\rho = 1 - \frac{288}{990} = 1. - 0.291 = 0.709$$

$$r = 2 \sin\left(\frac{\pi}{6}\rho\right) = 2 \sin 21.27^\circ = \underline{\underline{0.725}}$$

Chapter IX Mean Square Contingency

References:

(1) "On the Theory of Contingency and Its Relation to Association and Normal Correlation" by Karl Pearson Drapers' Company Research Memoirs, Biometric Series, I.
(2) "Statistical Methods Applied to Education" Harold O. Rugg P. 299 et seq.
(3) "Statistical Methods for Students in Education" Holzinger P. 273 et seq.
(4) "Introduction to The Theory of Statistics" Yule 64-67
(5) "Introduction to Mathematical Statistics" Carl J. West Ch. 13

I. Introduction

In the work with the coefficient of correlation, we were dealing with measured quantities, the statistics of variates. We now turn our attention to the relationship of traits which are not capable of quantitative measurement, the statistics of attributes.

A simple illustration will show the type of problem we are now to deal with. Suppose in a group of eighty-nine boys we wished to learn whether there was any association between their school work and their behavior and that these attributes could be tabulated as follows:

School Work	Behavior			
	Bad	Troublesome	Good	Excellent
Good	3	9	12	14
Medium	4	10	16	2
Poor	10	2	7	-

Clearly the product-moment method would not serve because we have no reasonable measurement for the various categories of behavior. We wish to find some method of measuring the amount of association which does not require us to determine scales for classifying the attributes.

This method has been developed by Karl Pearson in his coefficient of mean square contingency.

II. Contingency - Definition.

If we were considering the problem of the relationship of two attributes and classified them into a number of groups A_1, A_2, A_3 - - - - - A_s and B_1, B_2, B_3 - - - - - B_t, we would form a table containing s rows and t columns, or s x t compartments with the total frequency distributed into sub-groups corresponding to these compartments.

	B_1	B_2	B_3									B_t	
A_1													n_1
A_2													n_2
A_s													n_s
	m_1	m_2	m_3									m_t	N

If the total frequency were N and if the numbers falling in the groups A_1, A_2, etc. were n_1, n_2 - - - n_s, respectively (see table above) then the probability of one falling into one of these groups is $\frac{n_1}{N}$, $\frac{n_2}{N}$ - - - - - - $\frac{n_s}{N}$ respectively. In like manner if the number falling in the groups B_1, B_2 - - - B_t are M_1, M_2 - - - m_t respectively, the probability of one falling in one of these groups will be $\frac{m_1}{N}$, $\frac{m_2}{N}$, $\frac{m_3}{N}$, $\frac{m_4}{N}$, - - - $\frac{m_t}{N}$, respectively. Therefore, the number in the cell $A_r B_c$ to be expected on the theory of independent probability is

$$N \cdot \frac{n_r}{N} \cdot \frac{m_c}{N} = \frac{n_r m_c}{N}$$

for the probability that a measure will fall in a row A_r is $\frac{n_r}{N}$ and the probability that it will fall in a column B_c is $\frac{m_c}{N}$. Hence the probability that any one measure will fall in this row and column is $\frac{n_r m_c}{N^2}$, but there are N measures so the probability that any one will fall there is $N \cdot \frac{n_r}{N} \cdot \frac{m_c}{N}$

If the number actually observed in this cell is n_{rc} then

$$n_{rc} - \frac{n_r m_c}{N}$$ measures the deviation from independent probability of the measure falling in the compartment $A_r B_c$.

Pearson points out that the total deviation of the whole system from independent probability must be some function of $n_{rc} - \frac{n_r m_c}{N}$ for the whole table and he terms this total deviation from independent probability a measure of contingency. Therefore the greater the contingency, the greater must be the amount of correlation between the two attributes, for such a correlation is the measure of the degree of deviation from independence of occurrence. Pearson then points out that if we define

$$\chi^2 = \Sigma \left\{ \frac{\left(n_{rc} - \frac{n_r m_c}{N} \right)^2}{\frac{n_r m_c}{N}} \right\}$$

we will have a function of $n_{rc} - \frac{n_r m_c}{N}$ which will measure the degree of deviation of the series from independent probability and which will bring contingency into line with the customary notations of correlation. The formula used above is of the type developed by Elderton in "Frequency Curves and Correlation" on page 141 to measure the amount of agreement between two sets of figures. Here it is used to measure the amount of agreement between our observed data and the data of a table based on chance alone.

Definition: Mean Square Contingency.

Having defined χ^2, Pearson then defines ϕ^2, the mean square contingency $\phi^2 = \frac{\chi^2}{N}$

III. Development of the Formula for Mean Square Contingency.

$$C = \sqrt{\frac{\phi^2}{1+\phi^2}}$$

Let x and y denote the deviations from their respective means of two attributes, σ_x, σ_y are the standard deviations and r is the correlation. Then if the correlation table can be approximately represented by the normal correlation surface,

$$z_0 = \frac{N}{2\pi \sigma_x \sigma_y} e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right)}$$

represents the frequency with no correlation as previously discussed and

$$z = \frac{N}{2\pi\sqrt{1-r^2}\,\sigma_x \sigma_y} e^{-\frac{1}{2(1-r^2)}\left(\frac{x^2}{\sigma_x^2} - \frac{2rxy}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2}\right)}$$

represents the frequency with which we are dealing; i.e. the frequency of the observed data.

Since $\phi^2 = \dfrac{\chi^2}{N}$ and

$$\chi^2 = \Sigma \left\{ \frac{\left(n_{rc} - \frac{n_r m_c}{N}\right)^2}{\frac{n_r m_c}{N}} \right\}$$

then

$$\phi^2 = \frac{1}{N} \Sigma \left\{ \frac{\left(n_{rc} - \frac{n_r m_c}{N}\right)^2}{\frac{n_r m_c}{N}} \right\}$$

Therefore, if we sum over the entire table this reduces to

(1) $$\phi^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(z - z_0)^2}{N z_0}\, dx\, dy$$

(2) Substituting for z and z_0 and let $x' = \dfrac{x}{\sigma_x}$, $y' = \dfrac{y}{\sigma_y}$

$$\phi^2 = \frac{1}{2\pi}\left\{ \frac{1}{1-r^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(x'^2 \frac{1+r^2}{1-r^2} - \frac{4rx'y'}{1-r^2} + y'^2 \frac{1+r^2}{1-r^2}\right)} dx'\, dy' \right.$$

$$- \frac{2}{\sqrt{1-r^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left\{x'^2 \frac{1}{1-r^2} - \frac{2rx'y'}{1-r^2} + y'^2 \frac{1}{1-r^2}\right)} dx'\, dy'$$

$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x'^2 + y'^2)}\, dx'\, dy'$$

(3) $$\phi^2 = \frac{1}{1-r^2} \cdot \frac{1}{\sqrt{\left(\frac{1+r^2}{1-r^2}\right)^2 - \frac{4r^2}{(1-r^2)^2}}} - \frac{2}{\sqrt{1-r^2}} \cdot \frac{1}{\sqrt{\frac{1}{(1-r^2)^2} - \frac{r^2}{(1-r^2)^2}}} + 1$$

This follows from the fact that if $ac > b^2$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(ax^2 - 2bx + cy^2)}\, dx\, dy = \frac{2\pi}{\sqrt{ac-b^2}}$$

See Notes, Page 4

(4) Simplifying

$$\phi^2 = \frac{1}{1-r^2} - 2 + 1$$

(5) $$r = \pm \sqrt{\frac{\phi^2}{1+\phi^2}}$$

Some important conclusions can be drawn from this. Elderton P.148.

"(1) It shows clearly that r must lie between -1 and 1.

(2) Since the value of ϕ^2 will not be affected by the order of rows (or columns), it will be seen that it is permissable to interchange them, provided, of course, the whole column (or row) be moved at once.

(3) The proof shows that r will not necessarily be obtained exactly if a very small number of groups is used, because by using the integral calculus an infinite number of groups was assumed.

(4) We also assumed, however, that we were dealing with a perfectly smooth series; but since χ^2 is a measure of goodness of fit between the correlation and non-correlation figures, a very large number of groups gives undue prominence to chance deviation, due to the use of random sampling, and the value of r found from that of ϕ^2 may differ considerably from the value reached by the xy moment. Too fine a grouping may give a less accurate result than

a less fine one."

Pearson further points out that since ϕ^2 is a measure of deviation of the series from independent probability and therefore of the amount of association or correlation between the attributes involved, any function of this expression is also a proper measure. Therefore, in order to bring the coefficient of contingency into line with the notations used in the coefficient of correlation, he defines the coefficient of mean square contingency

$$C = \sqrt{\frac{\phi^2}{1+\phi^2}}$$

V. Necessity of Limiting the Use of the Coefficient of Contingency to 5 × 5 fold or Finer Classifications. Yule. P. 65-6.

$$C = \sqrt{\frac{\phi^2}{1+\phi^2}} = \sqrt{\frac{\lambda^2}{N+\lambda^2}}$$

Yule shows that coefficients when "calculated on different systems of classification are not comparable with each other. It is clearly de-desirable, for practical purposes, that two coefficients calculated from the same data, classified in two different ways, should be, at least approximately, identical. With the present coefficient this is not the case: if certain data be classified in, say (1) 6 × 6 fold, (2) 3 × 3 fold form, the coefficient in the latter form tends to be the least. The greatest possible value is, in fact, only unity if the number of classes be infinitely great; for any finite number of classes the limiting value of C is the smaller, the smaller the number of classes." Yule "Introduction to Theory of Statistics" P. 65.

The proof of this statement follows:

(1) $$\lambda^2 = \Sigma \left\{ \frac{\left(n_{rc} - \frac{n_r m_c}{N}\right)^2}{\frac{n_r m_c}{N}} \right\}$$

(2) $$\lambda^2 = \Sigma \left\{ \frac{(n_{rc})^2}{\frac{n_r m_c}{N}} \right\} - 2 \Sigma n_{rc} + \Sigma \frac{n_r m_c}{N}$$

(3) Let $$\Sigma \left\{ \frac{(n_{rc})^2}{\frac{n_r m_c}{N}} \right\} = S$$

Then $$\lambda^2 = S - 2N + N = S - N$$

(4) $$\therefore C = \sqrt{\frac{S-N}{S}}$$

Now suppose we are to deal with a t $\times$ t fold classification in which $n_r = m_r$ for all values of r; and suppose, further, that the association between the two attributes is perfect so $n_r m_r = n_r = m_r$ for all values of r, and the frequencies in the remaining cells are zero. The frequency is then concentrated in the diagonal compartments of the table. If we interpret our notation in the light of this hypothesis, we have:

$$n_{rc} = n_r = m_c \quad \text{or} \quad (n_{rc})^2 = n_r m_c$$

$$\therefore S = \Sigma \left\{ \frac{(n_{rc})^2}{\frac{n_r m_c}{N}} \right\} = \Sigma (N) = tN$$

So we may write

$$C = \sqrt{\frac{t-1}{t}}$$

This is the greatest value of C for a symmetrical t $\times$ t - fold classification.

Yule then shows for

t = 2	C cannot exceed	0.707
t = 3	" " "	0.816
t = 4	" " "	0.866
t = 5	" " "	0.894
t = 6	" " "	0.913
t = 7	" " "	0.926
t = 8	" " "	0.935
t = 9	" " "	0.943
t = 10	" " "	0.949

so that it is well to restrict the coefficient of contingency to 5×5 or finer classifications where the maximum value of C will at least approximate unity.

VI. Problem.

The coefficient of mean square contingency may be used for data quantitatively measured as well as for that which is qualitative. It may be used where one series is quantitative and one qualitative. The following example is from Rugg, P. 305, and shows the steps in using such a coefficient.

Relation Between Mental Age and Pedagogical Age.

		Mental Age							
		9	10	11	12	13	14	15	Totals
Pedagogical Age.	Retarded 2 years				2		7	2	11
	Retarded 1 year		1		4	9	3	1	18
	Normal			3	8	4	1		16
	Accelerated 1 year		5	10	6	2			23
	Accelerated 2 years	2	7	3	1	1			14
	Totals	2	13	16	21	16	11	3	N = 82

$$C = \sqrt{\frac{S-N}{S}}, \quad S = \Sigma \left\{ \frac{(n_{rc})^2}{\frac{n_r m_c}{N}} \right\}$$

Table giving $\dfrac{n_r m_c}{N}$

	Mental Age							
		9	10	11	12	13	14	15
Pedagogical Age.	Retarded 1 year				2.82		1.48	0.40
	Retarded 2 years		2.85		4.61	3.51	2.42	0.66
	Normal			3.12	4.10	3.12	2.15	
	Accelerated 1 year		3.65	4.49	5.89	4.49		
	Accelerated 2 years	0.34	2.22	2.73	3.59	2.73		

The 2.85 in circle above is arrived at by the following

$n_r = 18, \quad m_c = 13, \quad N = 82$

$$\frac{n_r m_c}{N} = \frac{13 \times 18}{82} = 2.85$$

Table Showing $\left\{ \dfrac{(n_{rc})^2}{\dfrac{n_r m_c}{N}} \right\}$

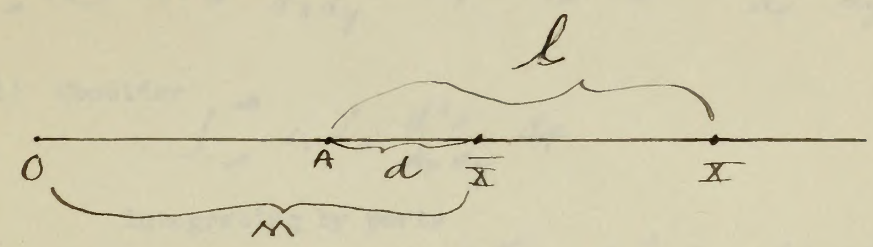
	Mental Age							
		9	10	11	12	13	14	15
Pedagogical Age.	Retarded 2 years				1.42		33.14	10
	Retarded 1 year		0.351		3.471	23.08	3.727	1.515
	Normal			2.88	15.61	5.13	0.465	
	Accelerated 1 year		6.85	22.27	6.11	0.891		
	Accelerated 2 years	11.735	22.07	3.295	0.279	0.367		

S = 174.656
N = 82
S-N = 92.656

$$C = \sqrt{\frac{92.656}{174.656}} = \sqrt{0.5305} = \underline{0.728}$$

Notes Page 1

I. Root Mean-Square Deviation



Definition of root mean-square

$$S^2 = \frac{1}{N} \Sigma (l^2)$$

or S^2 is the average of the squares of the deviations about an arbitrary origin A.

(1) $l = X - A$

(2) $M - A = d$

(3) $l = X + d - M$

(4) $l = X - M + d$

(5) Let $x = X - M$

(6) Then $l = x + d$

(7) $l^2 = x^2 + 2xd + d^2$

(8) $\Sigma(l^2) = \Sigma(x^2) + 2d\Sigma(x) + \Sigma(d^2)$

(9) $\Sigma(x)$ the sum of the deviations about the mean and equals zero.

(10) $\Sigma(l^2) = \Sigma(x^2) + \Sigma(d^2)$

(11) $\frac{\Sigma(l^2)}{N} = \frac{\Sigma(x^2)}{N} = \frac{\Sigma(d^2)}{N}$

(12) $S^2 = \sigma_x^2 + d^2$

Notes Page 2

II. To show

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} i_1 i_2 \frac{d^2 z}{dx\,dy}\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z \frac{di_1}{dx}\frac{di_2}{dy}\,dx\,dy.$$

(1) Consider

$$\int_{-\infty}^{\infty} i_1 i_2 \frac{d^2 z}{dx\,dy}\,dx$$

Integrating by parts

$$= i_1(x)\, i_2(y) \frac{dz}{dy}\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} i_2 \frac{dz}{dy}\frac{di_1}{dx}\,dx$$

$$= - \int_{-\infty}^{\infty} i_2 \frac{dz}{dy}\frac{di_1}{dx}\,dx$$

(2) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} i_1 i_2 \frac{d^2 z}{dx\,dy}\,dx\,dy = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} i_2 \frac{dz}{dy}\frac{di_1}{dx}\,dx\,dy$$

(3) Now consider $\int_{-\infty}^{\infty} i_2 \frac{dz}{dy}\frac{di_1}{dx}\,dy$

$$= \frac{di_1}{dx}\int_{-\infty}^{\infty} i_2 \frac{dz}{dy}\,dy$$

Integrating by parts

$$= \frac{di_1}{dx}\left\{\left[i_2(y)\, z\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} z \frac{di_2}{dy}\,dy\right\}$$

$$= -\frac{di_1}{dx}\int_{-\infty}^{\infty} z \frac{di_2}{dy}\,dy$$

(4) $$\therefore \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} i_1 i_2 \frac{d^2 z}{dx\,dy}\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z \frac{di_1}{dx}\frac{di_2}{dy}\,dx\,dy$$

Notes Page 3

III To show $\dfrac{d\,P_{g_1 g_2}}{dr} = \dfrac{N^3}{2\pi\sqrt{4-r^2}}$

(1) $$\frac{d\,P_{g_1 g_2}}{dr} = \sigma_1\sigma_2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z\,\frac{di_1}{dx}\,\frac{di_2}{dy}\,dx\,dy$$

$$i_1 = \frac{N}{\sqrt{2\pi}\,\sigma_1}\int_0^x e^{-\frac{1}{2}\frac{x^2}{\sigma_1^2}}\,dx \quad , \quad i_2 = \frac{N}{\sqrt{2\pi}\,\sigma_2}\int_0^y e^{-\frac{1}{2}\frac{y^2}{\sigma_2^2}}\,dy$$

$$\frac{di_1}{dx} = \frac{N}{\sqrt{2\pi}\,\sigma_1}\,e^{-\frac{1}{2}\frac{x^2}{\sigma_1^2}} \quad , \quad \frac{di_2}{dy} = \frac{N}{\sqrt{2\pi}\,\sigma_2}\,e^{-\frac{1}{2}\frac{y^2}{\sigma_2^2}}$$

Let $x = x'\sigma_1$, $y = y'\sigma_2$

$$\frac{di_1}{dx} = \frac{N}{\sqrt{2\pi}\,\sigma_1}\,e^{-\frac{1}{2(1-r^2)}(1-r^2)x'^2}$$

$$\frac{di_2}{dy} = \frac{N}{\sqrt{2\pi}\,\sigma_2}\,e^{-\frac{1}{2(1-r^2)}(1-r^2)y'^2}$$

$$z = \frac{N}{2\pi\sigma_1\sigma_2}\cdot\frac{1}{\sqrt{1-r^2}}\,e^{-\frac{1}{2(1-r^2)}(x'^2-2rx'y'+y'^2)}$$

(2) $$\frac{d\,P_{g_1 g_2}}{dr} = \frac{N^3}{4\pi^2\sqrt{1-r^2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2(1-r^2)}\left((2-r^2)x'^2-2rx'y'+(2-r^2)y'^2\right)}\,dx'dy'$$

(3) $$\frac{d\,P_{g_1 g_2}}{dr} = \frac{2\pi}{\sqrt{\left(\frac{2-r^2}{1-r^2}\right)^2-\frac{r^2}{(1-r^2)^2}}}\cdot\frac{N^3}{4\pi^2\sqrt{1-r^2}} = \frac{N^3}{2\pi\sqrt{4-r^2}}$$

for $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(ax^2-2bxy+cy^2)}\,dx\,dy = \frac{2\pi}{\sqrt{ac-b^2}}$$

if $ac > b^2$ See next page.

<u>Notes</u> Page 4

IV. To show

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(ax^2-2bxy+cy^2)}\,dx\,dy = \frac{2\pi}{\sqrt{ac-b^2}}$$

if $ac > b^2$

(1) The index may be written

$$-\frac{a}{2}\left(x-\frac{b}{a}y\right)^2 + \frac{y^2}{2a}(ac-b^2)$$

(2) Integrating with respect to x, since

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{h^2}}\,dx = h\sqrt{\pi}\text{, we have}$$

$$\sqrt{\frac{2\pi}{a}}\int_{-\infty}^{\infty} e^{-\frac{y^2}{2a}(ac-b^2)}\,dy$$

(3) Now integrating with respect to y, since

$$ac-b^2>0$$

we have $\sqrt{\dfrac{2\pi}{a}}\cdot\sqrt{\dfrac{2\pi a}{ac-b^2}} = \dfrac{2\pi}{\sqrt{ac-b^2}}$

Conclusion

In this paper I have tried to show something of the development of the formulas for simple correlation of three important types; the coefficient of correlation for linear regression, the most important in its frequent use; the coefficient of correlation from ranks; and the coefficient of mean square contingency. The first of these to be used where the data are represented by numerical measures and the method taking full account of the value and position of every measure in the series, the second to be used where only the positions of the measures are given, and the third when the data are not in terms of numerical measures but in the form of attributes. I have been interested in these because I felt they were suitable formulas for work usually done in statistics in Education. A more complete discussion should, of course, contain something of the correlation ratio to be used where the regression is non-linear, a study of tests for linearity, and a study of probable errors. These topics would form a suitable study in themselves.

In developing the coefficient of correlation by the correlation surface method certain assumptions based on the theory of probability were made, and the equation of the normal curve was used without deriving it. These could have been given a sound mathematical basis but it seemed wise to limit the paper and give references for their derivation.

In the chapter on correlation from ranks, the assumption was made that grades could be replaced by ranks. Pearson makes this assumption in the reference cited in that chapter. The sound method would be, it seems, to use grades exclusively but the work involved would be extremely laborious and the results not sufficiently different to warrant such an effort.

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Chapter X